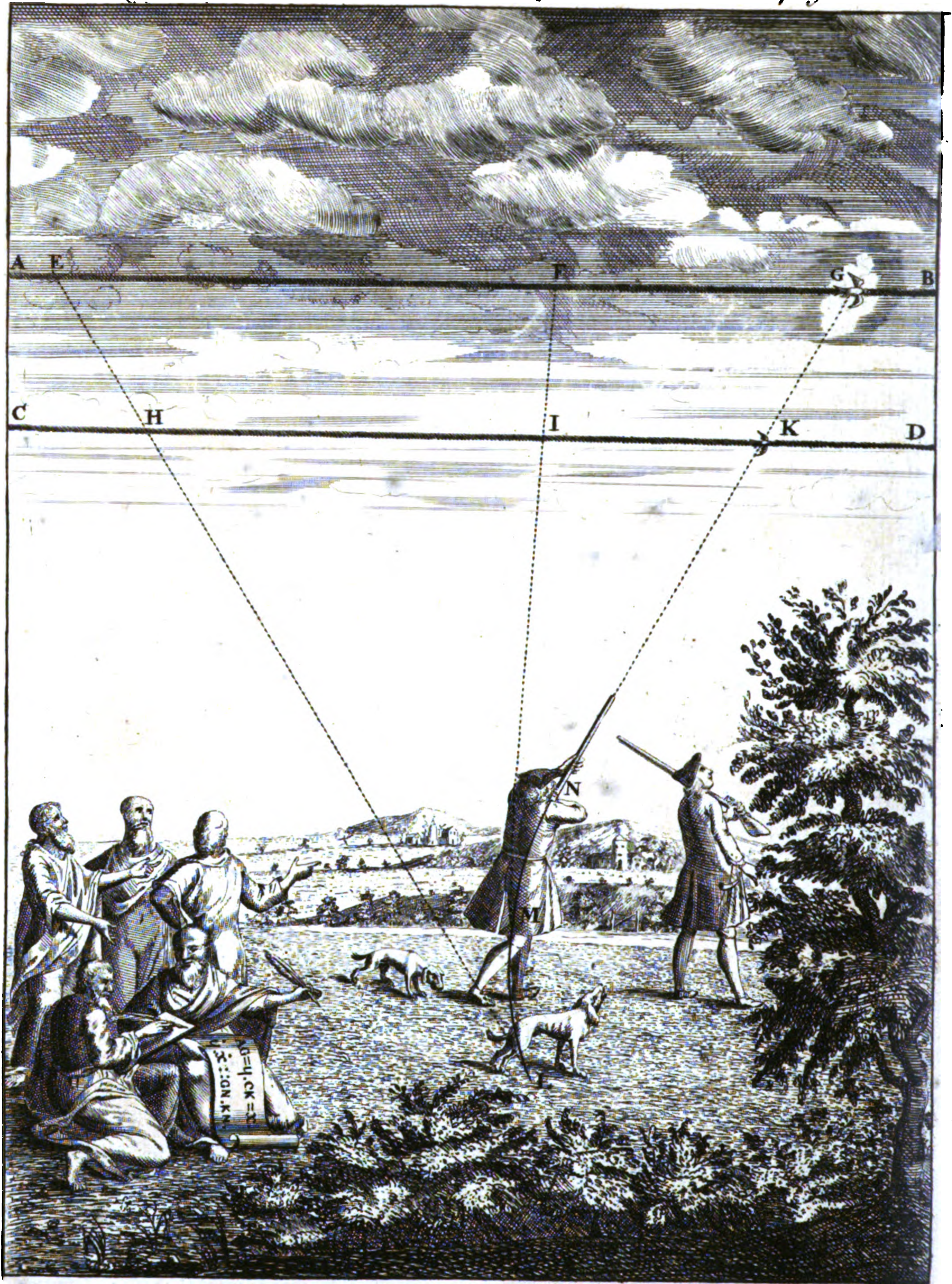
Sensibiles sensibilium velocitatum mensuræ. vid. pag. 273.



Τὰ κοινὰ καινῶς. τὰ καινὰ κοινῶς.

THE

METHOD of FLUXIONS

AND

INFINITE SERIES;

WITH ITS

Application to the Geometry of CURVE-LINES.

By the INVENTOR

Sir ISAAC NEWTON, Kt.

Late Preſident of the Royal Society.

Tranſlated from the AUTHOR's LATIN ORIGINAL
not yet made publick.

To which is ſubjoin'd,

A PERPETUAL COMMENT upon the whole Work,

Conſiſting of

ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,

In order to make this Treatiſe

A compleat Inſtitution for the uſe of LEARNERS.

By *JOHN COLSON*, M. A. and F. R. S.
Maſter of Sir *Joſeph Williamſon*'s free Mathematical-School at *Rocheſter*.

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TO

William Jones Eſq; F.R.S.

S I R,

IT was a laudable cuſtom among the ancient Geometers, and very worthy to be imitated by their Succeſſors, to addreſs their Mathematical labours, not ſo much to Men of eminent rank and ſtation in the world, as to Perſons of diſtinguiſh'd merit and proficience in the ſame Studies. For they knew very well, that ſuch only could be competent Judges of their Works, and would receive them with the eſteem they might deſerve. So far at leaſt I can copy after thoſe great Originals, as to chuſe a Patron for theſe Speculations, whoſe known skill and abilities in ſuch matters will enable him to judge, and whoſe known candor will incline him to judge favourably, of the ſhare I have had in the preſent performance. For as to the fundamental part of the Work, of which I am only the Interpreter, I know it cannot but pleaſe you; it will need no protection, nor can it receive a greater recommendation, than to bear the name of its illuſtrious Author. However, it very naturally applies itſelf to you, who had the honour (for I am ſure you think it ſo) of the Author's friendſhip and familiarity in his life-time; who had his own conſent to publiſh an elegant edition of ſome of his pieces, of a nature not very different from this; and who have ſo juſt an eſteem for, as well as knowledge of, his other moſt ſublime, moſt admirable, and juſtly celebrated Works.

But beſides theſe motives of a publick nature, I had others that more nearly concern myſelf. The many perſonal obligations I have received from you, and your generous manner of conferring them, require all the teſtimonies of gratitude in my power. Among the reſt, give me leave to mention one, (tho' it be a privilege I have enjoy'd in common with many others, who have the happineſs of your acquaintance,) which is, the free acceſs you have always allow'd me, to your copious Collection of whatever is choice and excellent in the Mathematicks. Your judgment and induſtry, in collecting thoſe valuable κειμήλια, are not more conſpicuous, than the freedom and readineſs with which you communicate them, to all ſuch who you know will apply them to their proper uſe, that is, to the general improvement of Science.

Before I take my leave, permit me, good Sir, to join my wiſhes to thoſe of the publick, that your own uſeful Lucubrations may ſee the light, with all convenient ſpeed; which, if I rightly conceive of them, will be an excellent methodical Introduction, not only to the mathematical Sciences in general, but alſo to theſe, as well as to the other curious and abſtruſe Speculations of our great Author. You are very well apprized, as all other good Judges muſt be, that to illuſtrate him is to cultivate real Science, and to make his Diſcoveries eaſy and familiar, will be no ſmall improvement in Mathematicks and Philoſophy.

That you will receive this addreſs with your uſual candor, and with that favour and friendſhip I have ſo long and often experienced, is the earneſt requeſt of,

SIR,

Your moſt obedient humble Servant,

J. COLSON.

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THE PREFACE.

I Cannot but very much congratulate with my Mathematical Readers, and think it one of the moſt fortunate circumſtances of my Life, that I have it in my power to preſent the publick with a moſt valuable Anecdote, of the greateſt Maſter in Mathematical and Philoſophical Knowledge, that ever appear'd in the World. And ſo much the more, becauſe this Anecdote is of an elementary nature, preparatory and introductory to his other moſt arduous and ſublime Speculations, and intended by himſelf for the inſtruction of Novices and Learners. I therefore gladly embraced the opportunity that was put into my hands, of publiſhing this poſthumous Work, becauſe I found it had been compoſed with that view and deſign. And that my own Country-men might firſt enjoy the benefit of this publication, I reſolved upon giving it in an *Engliſh* Tranſlation, with ſome additional Remarks of my own. I thought it highly injurious to the memory and reputation of the great Author, as well as invidious to the glory of our own Nation, that ſo curious and uſeful a piece ſhould be any longer ſuppreſs'd, and confined to a few private hands, which ought to be communicated to all the learned World for general Inſtruction. And more eſpecially at a time when the Principles of the Method here taught have been ſcrupulouſly ſifted and examin'd, have been vigorouſly oppoſed and (we may ſay) ignominiouſly rejected as inſufficient, by ſome Mathematical Gentlemen, who ſeem not to have derived their knowledge of them from their only true Source, that is, from our Author's own Treatiſe wrote expreſsly to explain them. And on the other hand, the Principles of this Method have been zealouſly and commendably defended by other Mathematical Gentlemen, who yet ſeem to have been as little acquainted with this Work, (or at leaſt to have over-look'd it,) the only genuine and original Fountain of this kind of knowledge. For what has been elſewhere deliver'd by our Author, concerning this Method, was only accidental and occaſional, and far from that copiouſneſs with which he treats of it here, and illuſtrates it with a great variety of choice Examples.

The learned and ingenious Dr. *Pemberton*, as he acquaints us in his View of Sir *Iſaac Newton*'s Philoſophy, had once a deſign of publiſhing this Work, with the conſent and under the inſpection of the Author himſelf; which if he had then accompliſh'd, he would certainly have deſerved and received the thanks of all lovers of Science. The Work would have then appear'd with a double advantage, as receiving the laſt Emendations of its great Author, and likewiſe in paſſing through the hands of ſo able an Editor. And among the other good effects of this publication, poſſibly it might have prevented all or a great part of thoſe Diſputes, which have ſince been raiſed, and which have been ſo ſtrenuouſly and warmly purſued on both ſides, concerning the validity of the Principles of this Method. They would doubtleſs have been placed in ſo good a light, as would have cleared them from any imputation of being in any wiſe defective, or not ſufficiently demonſtrated. But ſince the Author's Death, as the Doctor informs us, prevented the execution of that deſign, and ſince he has not thought fit to reſume it hitherto, it became needful that this publication ſhould be undertook by another, tho' a much inferior hand.

For it was now become highly neceſſary, that at laſt the great Sir *Iſaac* himſelf ſhould interpoſe, ſhould produce his genuine Method of Fluxions, and bring it to the teſt of all impartial and conſiderate Mathematicians; to ſhew its evidence and ſimplicity, to maintain and defend it in his own way, to convince his Opponents, and to teach his Diſciples and Followers upon what grounds they ſhould proceed in vindication of the Truth and Himſelf. And that this might be done the more eaſily and readily, I reſolved to accompany it with an ample Commentary, according to the beſt of my ſkill, and (I believe) according to the mind and intention of the Author, wherever I thought it needful; and particularly with an Eye to the fore-mention'd Controverſy. In which I have endeavour'd to obviate the difficulties that have been raiſed, and to explain every thing in ſo full a manner, as to remove all the objections of any force, that have been any where made, at leaſt ſuch as have occur'd to my obſervation. If what is here advanced, as there is good reaſon to hope, ſhall prove to the ſatiſfaction of thoſe Gentlemen, who firſt ſtarted theſe objections, and who (I am willing to ſuppoſe) had only the cauſe of Truth at heart; I ſhall be very glad to have contributed any thing, towards the removing of their Scruples. But if it ſhall happen otherwiſe, and what is here offer'd ſhould not appear to be ſufficient evidence, conviction, and demonſtration to them; yet I am perſuaded it will be ſuch to moſt other thinking Readers, who ſhall apply themſelves to it with unprejudiced and impartial minds; and then I ſhall not think my labour ill beſtow'd. It ſhould however be well conſider'd by thoſe Gentlemen, that the great number of Examples they will find here, to which the Method of Fluxions is ſucceſsfully apply'd, are ſo many vouchers for the truth of the Principles, on which that Method is founded. For the Deductions are always conformable to what has been derived from other uncontroverted Principles, and therefore muſt be acknowledg'd as true. This argument ſhould have its due weight, even with ſuch as cannot, as well as with ſuch as will not, enter into the proof of the Principles themſelves. And the *hypotheſis* that has been advanced to evade this concluſion, of one error in reaſoning being ſtill corrected by another equal and contrary to it, and that ſo regularly, conſtantly, and frequently, as it muſt be ſuppos'd to do here; this *hypotheſis*, I ſay, ought not to be ſeriouſly refuted, becauſe I can hardly think it is ſeriouſly propoſed.

The chief Principle, upon which the Method of Fluxions is here built, is this very ſimple one, taken from the Rational Mechanicks; which is, That Mathematical Quantity, particularly Extenſion, may be conceived as generated by continued local Motion; and that all Quantities whatever, at leaſt by analogy and accommodation, may be conceived as generated after a like manner. Conſequently there muſt be comparative Velocities of increaſe and decreaſe, during ſuch generations, whoſe Relations are fixt and determinable, and may therefore (problematically) be propoſed to be found. This Problem our Author here ſolves by the help of another Principle, not leſs evident; which ſuppoſes that Quantity is infinitely diviſible, or that it may (mentally at leaſt) ſo far continually diminiſh, as at laſt, before it is totally extinguiſh'd, to arrive at Quantities that may be call'd vaniſhing Quantities, or which are infinitely little, and leſs than any aſſignable Quantity. Or it ſuppoſes that we may form a Notion, not indeed of abſolute, but of relative and comparative infinity. 'Tis a very juſt exception to the Method of Indiviſibles, as alſo to the foreign infiniteſimal Method, that they have recourſe at once to infinitely little Quantities, and infinite orders and gradations of theſe, not relatively but abſolutely ſuch. They aſſume theſe Quantities *ſimul & ſemel*, without any ceremony, as Quantities that actually and obviouſly exiſt, and make Computations with them accordingly; the reſult of which muſt needs be as precarious, as the abſolute exiſtence of the Quantities they aſſume. And ſome late Geometricians have carry'd theſe Speculations, about real and abſolute Infinity, ſtill much farther, and have raiſed imaginary Syſtems of infinitely great and infinitely little Quantities, and their ſeveral orders and properties; which, to all ſober Inquirers into mathematical Truths, muſt certainly appear very notional and viſionary.

Theſe will be the inconveniencies that will ariſe, if we do not rightly diſtinguiſh between abſolute and relative Infinity. Abſolute Infinity, as ſuch, can hardly be the object either of our Conceptions or Calculations, but relative Infinity may, under a proper regulation. Our Author obſerves this diſtinction very ſtrictly, and introduces none but infinitely little Quantities that are relatively ſo; which he arrives at by beginning with finite Quantities, and proceeding by a gradual and neceſſary progreſs of diminution. His Computations always commence by finite and intelligible Quantities; and then at laſt he inquires what will be the reſult in certain circumſtances, when ſuch or ſuch Quantities are diminiſh'd *in infinitum*. This is a conſtant practice even in common Algebra and Geometry, and is no more than deſcending from a general Propoſition, to a particular Caſe which is certainly included in it. And from theſe eaſy Principles, managed with a vaſt deal of ſkill and ſagacity, he deduces his Method of Fluxions; which if we conſider only ſo far as he himſelf has carry'd it, together with the application he has made of it, either here or elſewhere, directly or indirectly, expreſly or tacitely, to the moſt curious Diſcoveries in Art and Nature, and to the ſublimeſt Theories: We may deſervedly eſteem it as the greateſt Work of Genius, and as the nobleſt Effort that ever was made by the Human Mind. Indeed it muſt be own'd, that many uſeful Improvements, and new Applications, have been ſince made by others, and probably will be ſtill made every day. For it is no mean excellence of this Method, that it is doubtleſs ſtill capable of a greater degree of perfection; and will always afford an inexhauſtible fund of curious matter, to reward the pains of the ingenious and induſtrious Analyſt.

As I am deſirous to make this as ſatisfactory as poſſible, eſpecially to the very learned and ingenious Author of the Diſcourſe call'd *The Analyſt*, whoſe eminent Talents I acknowledge myſelf to have a great

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great veneration for; I ſhall here endeavour to obviate ſome of his principal Objections to the Method of Fluxions, particularly ſuch as I have not touch'd upon in my Comment, which is ſoon to follow.

He thinks our Author has not proceeded in a demonſtrative and ſcientifical matter, in his *Princip. lib.* 2. *lem.* 2. where he deduces the Moment of a Rectangle, whoſe Sides are ſuppoſed to be variable Lines. I ſhall repreſent the matter Analytically thus, agreeably (I think) to the mind of the Author.

Let X and Y be two variable Lines, or Quantities, which at different periods of time acquire different values, by flowing or increaſing continually, either equably or alike inequably. For inſtance, let there be three periods of time, at which X becomes $A - \frac{1}{2}a$, A, and $A + \frac{1}{2}a$; and Y becomes $B - \frac{1}{2}b$, B, and $B + \frac{1}{2}b$ ſucceſſively and reſpectively; where A, *a*, B, *b*, are any quantities that may be aſſumed at pleaſure. Then at the ſame periods of time the variable Product or Rectangle XY will become $\overline{A - \frac{1}{2}a} \times \overline{B - \frac{1}{2}b}$, AB, and $\overline{A + \frac{1}{2}a} \times \overline{B + \frac{1}{2}b}$, that is, $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$, AB, and $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. Now in the interval from the firſt period of time to the ſecond, in which X from being $A - \frac{1}{2}a$ is become A, and in which Y from being $B - \frac{1}{2}b$ is become B, the Product XY from being $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$ becomes AB; that is, by Subtraction, its whole Increment during that interval is $\frac{1}{2}aB + \frac{1}{2}bA - \frac{1}{4}ab$. And in the interval from the ſecond period of time to the third, in which X from being A becomes $A + \frac{1}{2}a$, and in which Y from being B becomes $B + \frac{1}{2}b$, the Product XY from being AB becomes $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$; that is, by Subtraction, its whole Increment during that interval is $\frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. Add theſe two Increments together, and we ſhall have $aB + bA$ for the compleat Increment of the Product XY, during the whole interval of time, while X flow'd from the value $A - \frac{1}{2}a$ to $A + \frac{1}{2}a$, or Y flow'd from the value $B - \frac{1}{2}b$ to $B + \frac{1}{2}b$. Or it might have been found by one Operation, thus: While X flows from $A - \frac{1}{2}a$ to A, and thence to $A + \frac{1}{2}a$, or Y flows from $B - \frac{1}{2}b$ to B, and thence to $B + \frac{1}{2}b$, the Product XY will flow from $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$ to AB, and thence to $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$; therefore by Subtraction the whole Increment during that interval of time will be $aB + bA$. Q. E. D.

This may eaſily be illuſtrated by Numbers thus: Make A, *a*, B, *b*, equal to 9, 4, 15, 6, reſpectively; (or any other Numbers to be aſſumed at pleaſure.) Then the three ſucceſſive values of X will be 7, 9, 11, and the three ſucceſſive values of Y will be 12, 15, 18, reſpectively,

reſpectively. Alſo the three ſucceſſive values of the Product XY will be 84, 135, 198. But $aB + bA = 4 \times 15 + 6 \times 9 = 114 = 198 - 84$. Q. E. O.

Thus the Lemma will be true of any conceivable finite Increments whatever; and therefore by way of Corollary, it will be true of infinitely little Increments, which are call'd Moments, and which was the thing the Author principally intended here to demonſtrate. But in the caſe of Moments it is to be conſider'd, that X, or definitely $A - \frac{1}{2}a$, A, and $A + \frac{1}{2}a$, are to be taken indifferently for the ſame Quantity; as alſo Y, and definitely $B - \frac{1}{2}b$, B, $B + \frac{1}{2}b$. And the want of this Conſideration has occaſion'd not a few perplexities.

Now from hence the reſt of our Author's Concluſions, in the ſame Lemma, may be thus derived ſomething more explicitely. The Moment of the Rectangle AB being found to be $Ab + aB$, when the contemporary Moments of A and B are repreſented by a and b reſpectively; make $B = A$, and therefore $b = a$, and then the Moment of $A \times A$, or A^2, will be $Aa + aA$, or $2aA$. Again, make $B = A^2$, and therefore $b = 2aA$, and then the Moment of $A \times A^2$, or A^3, will be $2aA^2 + aA^2$, or $3aA^2$. Again, make $B = A^3$, and therefore $b = 3aA^2$, and then the Moment of $A \times A^3$, or A^4, will be $3aA^3 + aA^3$, or $4aA^3$. Again, make $B = A^4$, and therefore $b = 4aA^3$, and then the Moment of $A \times A^4$, or A^5, will be $4aA^4 + aA^4$, or $5aA^4$. And ſo on *in infinitum*. Therefore in general, aſſuming m to repreſent any integer affirmative Number, the Moment of A^m will be maA^{m-1}.

Now becauſe $A^m \times A^{-m} = 1$, (where m is any integer affirmative Number,) and becauſe the Moment of Unity, or any other conſtant quantity, is $= 0$; we ſhall have $A^m \times$ Mom. $A^{-m} + A^{-m} \times$ Mom. $A^m = 0$, or Mom. $A^{-m} = - A^{-2m} \times$ Mom. A^m. But Mom. $A^m = maA^{m-1}$, as found before; therefore Mom. $A^{-m} = - A^{-2m} \times maA^{m-1} = - maA^{-m-1}$. Therefore the Moment of A^m will be maA^{m-1}, when m is any integer Number, whether affirmative or negative.

And univerſally, if we put $A^{\frac{m}{n}} = B$, or $A^m = B^n$, where m and n may be any integer Numbers, affirmative or negative; then we ſhall have $maA^{m-1} = nbB^{n-1}$, or $b = \frac{maA^{m-1}}{nA^{\frac{mn-n}{n}}} = \frac{m}{n}aA^{\frac{m}{n}-1}$, which is the Moment of B, or of $A^{\frac{m}{n}}$. So that the Moment of A^m will be ſtill maA^{m-1}, whether *m* be affirmative or negative, integer or fraction.

The Moment of AB being $bA + aB$, and the Moment of CD being $dC + cD$; ſuppoſe $D = AB$, and therefore $d = bA + aB$, and then by Subſtitution the Moment of ABC will be $\overline{bA + aB} \times C + cAB = bAC + aBC + cAB$. And likewiſe the Moment of $A^m B^n$ will be $nbB^{n-1}A^m + maA^{m-1}B^n$. And ſo of any others.

Now there is ſo near a connexion between the Method of Moments and the Method of Fluxions, that it will be very eaſy to paſs from the one to the other. For the Fluxions or Velocities of increaſe, are always proportional to the contemporary Moments. Thus if for A, B, C, *&c.* we write x, y, z, &c. for a, b, c, &c. we may write $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. Then the Fluxion of xy will be $\dot{x}y + x\dot{y}$, the Fluxion of x^m will be $m\dot{x}x^{m-1}$, whether *m* be integer or fraction, affirmative or negative; the Fluxion of xyz will be $\dot{x}yz + x\dot{y}z + xy\dot{z}$, and the Fluxion of $x^m y^n$ will be $m\dot{x}x^{m-1}y^n + nx^m\dot{y}y^{n-1}$. And ſo of the reſt.

Or the former Inquiry may be placed in another view, thus: Let A and $A + a$ be two ſucceſſive values of the variable Quantity X, as alſo B and $B + b$ be two ſucceſſive and contemporary values of Y; then will AB and $AB + aB + bA + ab$ be two ſucceſſive and contemporary values of the variable Product XY. And while X, by increaſing perpetually, flows from its value A to $A + a$, or Y flows from B to $B + b$; XY at the ſame time will flow from AB to $AB + aB + bA + ab$, during which time its whole Increment, as appears by Subtraction, will become $aB + bA + ab$. Or in Numbers thus: Let A, a, B, b, be equal to 7, 4, 12, 6, reſpectively; then will the two ſucceſſive values of X be 7, 11, and the two ſucceſſive values of Y will be 12, 18. Alſo the two ſucceſſive values of the Product XY will be 84, 198. But the Increment $aB + bA + ab = 48 + 42 + 24 = 114 = 198 - 84$, as before.

And thus it will be as to all finite Increments: But when the Increments become Moments, that is, when a and b are ſo far diminiſh'd, as to become infinitely leſs than A and B; at the ſame time ab will become infinitely leſs than either aB or bA, (for $aB . ab :: B . b$, and $bA . ab :: A . a$,) and therefore it will vaniſh in reſpect of them. In which caſe the Moment of the Product or Rectangle will be $aB + bA$, as before. This perhaps is the more obvious and direct way of proceeding, in the preſent Inquiry; but, as there was room for choice, our Author thought fit to chuſe the former way, as the more elegant, and in which he was under no neceſſity of having recourſe to that Principle, that quantities ariſing in an Equation, which are infinitely leſs than the others, may be neglected or expunged in compariſon of thoſe others. Now to avoid the uſe of this Principle, tho' otherwiſe a true one, was all the Artifice uſed on this occaſion, which certainly was a very fair and juſtifiable one.

I ſhall conclude my Obſervations with conſidering and obviating the Objections that have been made, to the uſual Method of finding the Increment, Moment, or Fluxion of any indefinite power x^n of the variable quantity x, by giving that Inveſtigation in ſuch a manner, as to leave (I think) no room for any juſt exceptions to it. And the rather becauſe this is a leading point, and has been ſtrangely perverted and miſrepreſented.

In order to find the Increment of the variable quantity or power x^n, (or rather its relation to the Increment of x, conſider'd as given; becauſe Increments and Moments can be known only by compariſon with other Increments and Moments, as alſo Fluxions by compariſon with other Fluxions;) let us make $x^n = y$, and let X and Y be any ſynchronous Augments of x and y. Then by the hypotheſis we ſhall have the Equation $\overline{x+X}|^n = y+Y$; for in any Equation the variable Quantities may always be increaſed by their ſynchronous Augments, and yet the Equation will ſtill hold good. Then by our Author's famous Binomial Theorem we ſhall have $y+Y = x^n + nx^{n-1}X + n\times\frac{n-1}{2}x^{n-2}X^2 + n\times\frac{n-1}{2}\times\frac{n-2}{3}x^{n-3}X^3$, &c. or removing the equal Quantities y and x^n, it will be $Y = nx^{n-1}X + n\times\frac{n-1}{2}x^{n-2}X^2 + n\times\frac{n-1}{2}\times\frac{n-2}{3}x^{n-3}X^3$, &c. So that when X denotes the given Increment of the variable quantity x, Y will here denote the ſynchronous Increment of the indefinite power y or x^n; whoſe value therefore, in all caſes, may be had from this Series. Now that we may be ſure we proceed regularly, we will verify this thus far, by a particular and familiar inſtance or two. Suppoſe $n = 2$, then $Y = 2xX + X^2$. That is, while x flows or increaſes to $x+X$, x^2 in the ſame time, by its Increment $Y = 2xX + X^2$, will increaſe to $x^2 + 2xX + X^2$, which we otherwiſe know to be true. Again, ſuppoſe $n = 3$, then $Y = 3x^2X + 3xX^2 + X^3$. Or while x increaſes to $x+X$, x^3 by its Increment $Y = 3x^2X + 3xX^2 + X^3$ will increaſe to $x^3 + 3x^2X + 3xX^2 + X^3$. And ſo in all other particular caſes, whereby we may plainly perceive, that this general Concluſion muſt be certain and indubitable.

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This Series therefore will be always true, let the Augments X and Y be ever ſo great, or ever ſo little; for the truth does not at all depend on the circumſtance of their magnitude. Nay, when they are infinitely little, or when they become Moments, it muſt be true alſo, by virtue of the general Concluſion. But when X and Y are diminiſh'd *in infinitum,* ſo as to become at laſt infinitely little, the greater powers of X muſt needs vaniſh firſt, as being relatively of an infinitely leſs value than the ſmaller powers. So that when they are all expunged, we ſhall neceſſarily obtain the Equation $Y = nx^{n-1}X$; where the remaining Terms are likewiſe infinitely little, and conſequently would vaniſh, if there were other Terms in the Equation, which were (relatively) infinitely greater than themſelves. But as there are not, we may ſecurely retain this Equation, as having an undoubted right ſo to do; and eſpecially as it gives us an uſeful piece of information, that X and Y, tho' themſelves infinitely little, or vaniſhing quantities, yet they vaniſh in proportion to each other as 1 to nx^{n-1}. We have therefore learn'd at laſt, that the Moment by which x increaſes, or X, is to the contemporary Moment by which x^n increaſes, or Y, as 1 is to nx^{n-1}. And their Fluxions, or Velocities of increaſe, being in the ſame proportion as their ſynchronous Moments, we ſhall have $nx^{n-1}\dot{x}$ for the Fluxion of x^n, when the Fluxion of x is denoted by $\dot{x}$.

I cannot conceive there can be any pretence to inſinuate here, that any unfair artifices, any leger-de-main tricks, or any ſhifting of the hypotheſis, that have been ſo ſeverely complain'd of, are at all made uſe of in this Inveſtigation. We have legitimately derived this general Concluſion in finite Quantities, that in all caſes the relation of the Increments will be $Y = nx^{n-1}X + n \times \frac{n-1}{2}x^{n-2}X^2$, &c. of which one particular caſe is, when X and Y are ſuppoſed continually to decreaſe, till they finally terminate in nothing. But by thus continually decreaſing, they approach nearer and nearer to the Ratio of 1 to nx^{n-1}, which they attain to at the very inſtant of their vaniſhing, and not before. This therefore is their ultimate Ratio, the Ratio of their Moments, Fluxions, or Velocities, by which x and x^n continually increaſe or decreaſe. Now to argue from a general Theorem to a particular caſe contain'd under it, is certainly one of the moſt legitimate and logical, as well as one of the moſt uſual and uſeful ways of arguing, in the whole compaſs of the Mathematicks. To object here, that after we have made X and Y to ſtand for ſome quantity, we are not at liberty to make them nothing, or no quantity, or vaniſhing quantities, is not an Objection againſt the

Method of Fluxions, but againſt the common Analyticks. This Method only adopts this way of arguing, as a conſtant practice in the vulgar Algebra, and refers us thither for the proof of it. If we have an Equation any how compos'd of the general Numbers *a*, *b*, *c*, &c. it has always been taught, that we may interpret theſe by any particular Numbers at pleaſure, or even by 0, provided that the Equation, or the Conditions of the Queſtion, do not expreſsly require the contrary. For general Numbers, as ſuch, may ſtand for any definite Numbers in the whole Numerical Scale; which Scale (I think) may be thus commodiouſly repreſented, &c. — 3, — 2, — 1, 0, 1, 2, 3, 4, &c. where all poſſible fractional Numbers, intermediate to theſe here expreſs'd, are to be conceived as interpolated. But in this Scale the Term 0 is as much a Term or Number as any other, and has its analogous properties in common with the reſt. We are likewiſe told, that we may not give ſuch values to general Symbols afterwards, as they could not receive at firſt; which if admitted is, I think, nothing to the preſent purpoſe. It is always moſt eaſy and natural, as well as moſt regular, inſtructive, and elegant, to make our Inquiries as much in general Terms as may be, and to deſcend to particular caſes by degrees, when the Problem is nearly brought to a concluſion. But this is a point of convenience only, and not a point of neceſſity. Thus in the preſent caſe, inſtead of deſcending from finite Increments to infinitely little Moments, or vaniſhing Quantities, we might begin our Computation with thoſe Moments themſelves, and yet we ſhould arrive at the ſame Concluſions. As a proof of which we may conſult our Author's own Demonſtration of his Method, in pag. 24. of this Treatiſe. In ſhort, to require this is juſt the ſame thing as to inſiſt, that a Problem, which naturally belongs to Algebra, ſhould be ſolved by common Arithmetick; which tho' poſſible to be done, by purſuing backwards all the ſteps of the general proceſs, yet would be very troubleſome and operoſe, and not ſo inſtructive, or according to the true Rules of Art.

But I am apt to ſuſpect, that all our doubts and ſcruples about Mathematical Inferences and Argumentations, eſpecially when we are ſatisfied that they have been juſtly and legitimately conducted, may be ultimately reſolved into a ſpecies of infidelity and diſtruſt. Not in reſpect of any implicite faith we ought to repoſe on meer human authority, tho' ever ſo great, (for that, in Mathematicks, we ſhould utterly diſclaim,) but in reſpect of the Science itſelf. We are hardly brought to believe, that the Science is ſo perfectly regular and uniform,

form, ſo infinitely conſiſtent, conſtant, and accurate, as we ſhall really find it to be, when after long experience and reflexion we ſhall have overcome this prejudice, and ſhall learn to purſue it rightly. We do not readily admit, or eaſily comprehend, that Quantities have an infinite number of curious and ſubtile properties, ſome near and obvious, others remote and abſtruſe, which are all link'd together by a neceſſary connexion, or by a perpetual chain, and are then only diſcoverable when regularly and cloſely purſued; and require our truſt and confidence in the Science, as well as our induſtry, application, and obſtinate perſeverance, our ſagacity and penetration, in order to their being brought into full light. That Nature is ever conſiſtent with herſelf, and never proceeds in theſe Speculations *per ſaltum*, or at random, but is infinitely ſcrupulous and ſolicitous, as we may ſay, in adhering to Rule and Analogy. That whenever we make any regular Poſitions, and purſue them through ever ſo great a variety of Operations, according to the ſtrict Rules of Art; we ſhall always proceed through a ſeries of regular and well-connected tranſmutations, (if we would but attend to 'em,) till at laſt we arrive at regular and juſt Concluſions. That no properties of Quantity are intirely deſtructible, or are totally loſt and aboliſh'd, even tho' proſecuted to infinity itſelf; for if we ſuppoſe ſome Quantities to become infinitely great, or infinitely little, or nothing, or leſs than nothing, yet other Quantities that have a certain relation to them will only undergo proportional, and often finite alterations, will ſympathize with them, and conform to 'em in all their changes; and will always preſerve their analogical nature, form, or magnitude, which will be faithfully exhibited and diſcover'd by the reſult. This we may collect from a great variety of Mathematical Speculations, and more particularly when we adapt Geometry to Analyticks, and Curve-lines to Algebraical Equations. That when we purſue general Inquiries, Nature is infinitely prolifick in particulars that will reſult from them, whether in a direct ſubordination, or whether they branch out collaterally; or even in particular Problems, we may often perceive that theſe are only certain caſes of ſomething more general, and may afford good hints and aſſiſtances to a ſagacious Analyſt, for aſcending gradually to higher and higher Diſquiſitions, which may be proſecuted more univerſally than was at firſt expected or intended. Theſe are ſome of thoſe Mathematical Principles, of a higher order, which we find a difficulty to admit, and which we ſhall never be fully convinced of, or know the whole uſe of, but from much practice and attentive conſideration; but more eſpecially by a diligent peruſal, and cloſe examination, of this and the other Works of our illuſtrious Author. He abounded in theſe ſublime views and inquiries, had acquired an accurate and habitual knowledge of all theſe, and of many more general Laws, or Mathematical Principles of a ſuperior kind, which may not improperly be call'd *The Philoſophy of Quantity*; and which, aſſiſted by his great Genius and Sagacity, together with his great natural application, enabled him to become ſo compleat a Maſter in the higher Geometry, and particularly in the Art of Invention. This Art, which he poſſeſt in the greateſt perfection imaginable, is indeed the ſublimeſt, as well as the moſt difficult of all Arts, if it properly may be call'd ſuch; as not being reducible to any certain Rules, nor can be deliver'd by any Precepts, but is wholly owing to a happy ſagacity, or rather to a kind of divine Enthuſiaſm. To improve Inventions already made, to carry them on, when begun, to farther perfection, is certainly a very uſeful and excellent Talent; but however is far inferior to the Art of Diſcovery, as having a πῦ ςῶ, or certain *data* to proceed upon, and where juſt method, cloſe reaſoning, ſtrict attention, and the Rules of Analogy, may do very much. But to ſtrike out new lights, to adventure where no footſteps had ever been ſet before, *nullius antè trita ſolo*; this is the nobleſt Endowment that a human Mind is capable of, is reſerved for the choſen few *quos Jupiter æquus amavit*, and was the peculiar and diſtinguiſhing Character of our great Mathematical Philoſopher. He had acquired a compleat knowledge of the Philoſophy of Quantity, or of its moſt eſſential and moſt general Laws; had conſider'd it in all views, had purſued it through all its diſguiſes, and had traced it through all its Labyrinths and Receſſes; in a word, it may be ſaid of him not improperly, that he tortured and tormented Quantities all poſſible ways, to make them confeſs their Secrets, and diſcover their Properties.

The Method of Fluxions, as it is here deliver'd in this Treatiſe, is a very pregnant and remarkable inſtance of all theſe particulars: To take a curſory view of which, we may conveniently enough divide it into theſe three parts. The firſt will be the Introduction, or the Method of infinite Series. The ſecond is the Method of Fluxions, properly ſo call'd. The third is the application of both theſe Methods to ſome very general and curious Speculations, chiefly in the Geometry of Curve-lines.

As to the firſt, which is the Method of infinite Series, in this the Author opens a new kind of Arithmetick, (new at leaſt at the time of his writing this,) or rather he vaſtly improves the old. For

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he extends the received Notation, making it compleatly univerſal, and ſhews, that as our common Arithmetick of Integers received a great Improvement by the introduction of decimal Fractions; ſo the common Algebra or Analyticks, as an univerſal Arithmetick, will receive a like Improvement by the admiſſion of his Doctrine of infinite Series, by which the ſame analogy will be ſtill carry'd on, and farther advanced towards perfection. Then he ſhews how all complicate Algebraical Expreſſions may be reduced to ſuch Series, as will continually converge to the true values of thoſe complex quantities, or their Roots, and may therefore be uſed in their ſtead: whether thoſe quantities are Fractions having multinomial Denominators, which are therefore to be reſolved into ſimple Terms by a perpetual Diviſion; or whether they are Roots of pure Powers, or of affected Equations, which are therefore to be reſolved by a perpetual Extraction. And by the way, he teaches us a very general and commodious Method for extracting the Roots of affected Equations in Numbers. And this is chiefly the ſubſtance of his Method of infinite Series.

The Method of Fluxions comes next to be deliver'd, which indeed is principally intended, and to which the other is only preparatory and ſubſervient. Here the Author diſplays his whole ſkill, and ſhews the great extent of his Genius. The chief difficulties of this he reduces to the Solution of two Problems, belonging to the abſtract or Rational Mechanicks. For the direct Method of Fluxions, as it is now call'd, amounts to this Mechanical Problem, *The length of the Space deſcribed being continually given, to find the Velocity of the Motion at any time propoſed.* Alſo the inverſe Method of Fluxions has, for a foundation, the Reverſe of this Problem, which is, *The Velocity of the Motion being continually given, to find the Space deſcribed at any time propoſed.* So that upon the compleat Analytical or Geometrical Solution of theſe two Problems, in all their varieties, he builds his whole Method.

His firſt Problem, which is, *The relation of the flowing Quantities being given, to determine the relation of their Fluxions,* he diſpatches very generally. He does not propoſe this, as is uſually done, *A flowing Quantity being given, to find its Fluxion*; for this gives us too lax and vague an Idea of the thing, and does not ſufficiently ſhew that Compariſon, which is here always to be underſtood. Fluents and Fluxions are things of a relative nature, and ſuppoſe two at leaſt, whoſe relation or relations ſhould always be expreſs'd by Equations. He requires therefore that all ſhould be reduced to Equations, by which the relation of the flowing Quantities will be exhibited, and their comparative magnitudes will be more eaſily eſtimated; as alſo the comparative magnitudes of their Fluxions. And beſides, by this means he has an opportunity of reſolving the Problem much more generally than is commonly done. For in the uſual way of taking Fluxions, we are confined to the Indices of the Powers, which are to be made Coefficients; whereas the Problem in its full extent will allow us to take any Arithmetical Progreſſions whatever. By this means we may have an infinite variety of Solutions, which tho' different in form, will yet all agree in the main; and we may always chuſe the ſimpleſt, or that which will beſt ſerve the preſent purpoſe. He ſhews alſo how the given Equation may comprehend ſeveral variable Quantities, and by that means the Fluxional Equation may be found, notwithſtanding any ſurd quantities that may occur, or even any other quantities that are irreducible, or Geometrically irrational. And all this is derived and demonſtrated from the properties of Moments. He does not here proceed to ſecond, or higher Orders of Fluxions, for a reaſon which will be aſſign'd in another place.

His next Problem is, *An Equation being propoſed exhibiting the relation of the Fluxions of Quantities, to find the relation of thoſe Quantities, or Fluents, to one another*; which is the direct Converſe of the foregoing Problem. This indeed is an operoſe and difficult Problem, taking it in its full extent, and requires all our Author's ſkill and addreſs; which yet he ſolves very generally, chiefly by the aſſiſtance of his Method of infinite Series. He firſt teaches how we may return from the Fluxional Equation given, to its correſponding finite Fluential or Algebraical Equation, when that can be done. But when it cannot be done, or when there is no ſuch finite Algebraical Equation, as is moſt commonly the caſe, yet however he finds the Root of that Equation by an infinite converging Series, which anſwers the ſame purpoſe. And often he ſhews how to find the Root, or Fluent required, by an infinite number of ſuch Series. His proceſſes for extracting theſe Roots are peculiar to himſelf, and always contrived with much ſubtilty and ingenuity.

The reſt of his Problems are an application or an exemplification of the foregoing. As when he determines the *Maxima* and *Minima* of quantities in all caſes. When he ſhews the Method of drawing Tangents to Curves, whether Geometrical or Mechanical; or however the nature of the Curve may be defined, or refer'd to right Lines or other Curves. Then he ſhews how to find the Center or Radius of Curvature, of any Curve whatever, and that in a ſimple but general manner; which he illuſtrates by many curious Examples, and purſues many other ingenious Problems, that offer themſelves by the way. After which he diſcuſſes another very ſubtile and intirely new Problem about Curves, which is, to determine the quality of the Curvity of any Curve, or how its Curvature varies in its progreſs through the different parts, in reſpect of equability or inequability.

He then applies himſelf to conſider the Areas of Curves; and ſhews us how we may find as many Quadrable Curves as we pleaſe, or ſuch whoſe Areas may be compared with thoſe of right-lined Figures. Then he teaches us to find as many Curves as we pleaſe, whoſe Areas may be compared with that of the Circle, or of the Hyperbola, or of any other Curve that ſhall be aſſign'd; which he extends to Mechanical as well as Geometrical Curves. He then determines the Area in general of any Curve that may be propoſed, chiefly by the help of infinite Series; and gives many uſeful Rules for aſcertaining the Limits of ſuch Areas. And by the way he ſquares the Circle and Hyperbola, and applies the Quadrature of this to the conſtructing of a Canon of Logarithms. But chiefly he collects very general and uſeful Tables of Quadratures, for readily finding the Areas of Curves, or for comparing them with the Areas of the Conic Sections; which Tables are the ſame as thoſe he has publiſh'd himſelf, in his Treatiſe of Quadratures. The uſe and application of theſe he ſhews in an ample manner, and derives from them many curious Geometrical Conſtructions, with their Demonſtrations.

Laſtly, he applies himſelf to the Rectification of Curves, and ſhews us how we may find as many Curves as we pleaſe, whoſe Curve-lines are capable of Rectification; or whoſe Curve-lines, as to length, may be compared with the Curve-lines of any Curves that ſhall be aſſign'd. And concludes in general, with rectifying any Curve-lines that may be propoſed, either by the aſſiſtance of his Tables of Quadratures, when that can be done, or however by infinite Series. And this is chiefly the ſubſtance of the preſent Work. As to the account that perhaps may be expected, of what I have added in my Annotations; I ſhall refer the inquiſitive Reader to the Preface, which will go before that part of the Work.

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THE CONTENTS.

THE Introduction, or the Method of reſolving complex Quantities into infinite Series of ſimple Terms. —— pag. 1

Prob. 1. *From the given Fluents to find the Fluxions.* —— p. 21

Prob. 2. *From the given Fluxions to find the Fluents.* —— p. 25

Prob. 3. *To determine the* Maxima *and* Minima *of Quantities.* p. 44

Prob. 4. *To draw Tangents to Curves.* —— p. 46

Prob. 5. *To find the Quantity of Curvature in any Curve.* p. 59

Prob. 6. *To find the Quality of Curvature in any Curve.* p. 75

Prob. 7. *To find any number of Quadrable Curves.* p. 80

Prob. 8. *To find Curves whoſe Areas may be compared to thoſe of the Conic Sections.* —— p. 81

Prob. 9. *To find the Quadrature of any Curve aſſign'd.* p. 86

Prob. 10. *To find any number of rectifiable Curves.* p. 124

Prob. 11. *To find Curves whoſe Lines may be compared with any Curve-lines aſſign'd.* —— p. 129

Prob. 12. *To rectify any Curve-lines aſſign'd.* —— p. 134

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THE
METHOD of FLUXIONS,
AND
INFINITE SERIES.

INTRODUCTION: *Or, the Reſolution of Equations by Infinite Series.*

1. HAVING obſerved that moſt of our modern Geometricians, neglecting the Synthetical Method of the Ancients, have apply'd themſelves chiefly to the cultivating of the Analytical Art; by the aſſiſtance of which they have been able to overcome ſo many and ſo great difficulties, that they ſeem to have exhauſted all the Speculations of Geometry, excepting the Quadrature of Curves, and ſome other matters of a like nature, not yet intirely diſcuſs'd: I thought it not amiſs, for the ſake of young Students in this Science, to compoſe the following Treatiſe, in which I have endeavour'd to enlarge the Boundaries of Analyticks, and to improve the Doctrine of Curve-lines.

2. Since there is a great conformity between the Operations in Species, and the ſame Operations in common Numbers; nor do they ſeem to differ, except in the Characters by which they are re-

preſented, the firſt being general and indefinite, and the other definite and particular: I cannot but wonder that no body has thought of accommodating the lately-diſcover'd Doctrine of Decimal Fractions in like manner to Species, (unleſs you will except the Quadrature of the Hyberbola by Mr. *Nicolas Mercator*;) eſpecially ſince it might have open'd a way to more abſtruſe Diſcoveries. But ſince this Doctrine of Species, has the ſame relation to Algebra, as the Doctrine of Decimal Numbers has to common Arithmetick; the Operations of Addition, Subtraction, Multiplication, Diviſion, and Extraction of Roots, may eaſily be learned from thence, if the Learner be but ſkill'd in Decimal Arithmetick, and the Vulgar Algebra, and obſerves the correſpondence that obtains between Decimal Fractions and Algebraick Terms infinitely continued. For as in Numbers, the Places towards the right-hand continually decreaſe in a Decimal or Subdecuple Proportion; ſo it is in Species reſpectively, when the Terms are diſpoſed, (as is often enjoin'd in what follows,) in an uniform Progreſſion infinitely continued, according to the Order of the Dimenſions of any Numerator or Denominator. And as the convenience of Decimals is this, that all vulgar Fractions and Radicals, being reduced to them, in ſome meaſure acquire the nature of Integers, and may be managed as ſuch; ſo it is a convenience attending infinite Series in Species, that all kinds of complicate Terms, (ſuch as Fractions whoſe Denominators are compound Quantities, the Roots of compound Quantities, or of affected Equations, and the like,) may be reduced to the Claſs of ſimple Quantities; that is, to an infinite Series of Fractions, whoſe Numerators and Denominators are ſimple Terms; which will no longer labour under thoſe difficulties, that in the other form ſeem'd almoſt inſuperable. Firſt therefore I ſhall ſhew how theſe Reductions are to be perform'd, or how any compound Quantities may be reduced to ſuch ſimple Terms, eſpecially when the Methods of computing are not obvious. Then I ſhall apply this Analyſis to the Solution of Problems.

3. Reduction by Diviſion and Extraction of Roots will be plain from the following Examples, when you compare like Methods of Operation in Decimal and in Specious Arithmetick.

Examples of Reduction by Diviſion.

4. The Fraction $\frac{aa}{b+x}$ being propoſed, divide aa by $b+x$ in the following manner:

$$\begin{array}{l} b+x)\,aa+0\,\left(\frac{aa}{b}-\frac{aax}{b^2}+\frac{aax^2}{b^3}-\frac{aax^3}{b^4}+\frac{aax^4}{b^5},\ \&c.\right. \\ \qquad\quad \underline{aa+\frac{aax}{b}} \\ \qquad\quad 0-\frac{aax}{b}+0 \\ \qquad\quad \underline{-\frac{aax}{b}-\frac{aax^2}{b^2}} \\ \qquad\qquad\quad 0+\frac{a^2x^2}{b^2}+0 \\ \qquad\qquad\quad \underline{+\frac{a^2x^2}{b^2}+\frac{a^2x^3}{b^3}} \\ \qquad\qquad\qquad 0-\frac{a^2x^3}{b^3}+0 \\ \qquad\qquad\qquad \underline{-\frac{a^2x^3}{b^3}-\frac{a^2x^4}{b^4}} \\ \qquad\qquad\qquad\quad 0+\frac{a^2x^4}{b^4}\ \&c. \end{array}$$

The Quotient therefore is $\frac{aa}{b}-\frac{a^2x}{b^2}+\frac{a^2x^2}{b^3}-\frac{a^2x^3}{b^4}+\frac{a^2x^4}{b^5}$, &c. which Series, being infinitely continued, will be equivalent to $\frac{aa}{b+x}$. Or making x the firſt Term of the Diviſor, in this manner, $x+b)\,aa+0\,($the Quotient will be $\frac{aa}{x}-\frac{aab}{x^2}+\frac{aab^2}{x^3}-\frac{a^2b^3}{x^4}$, &c. found as by the foregoing Proceſs.

5. In like manner the Fraction $\frac{1}{1+xx}$ will be reduced to $1-x^2+x^4-x^6+x^8$, &c. or to $x^{-2}-x^{-4}+x^{-6}-x^{-8}$, &c.

6. And the Fraction $\frac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}$ will be reduced to $2x^{\frac{1}{2}}-2x+7x^{\frac{3}{2}}-13x^2+34x^{\frac{5}{2}}$, &c.

7. Here it will be proper to obſerve, that I make uſe of x^{-1}, x^{-2}, x^{-3}, x^{-4}, &c. for $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$, $\frac{1}{x^4}$, &c. of $x^{\frac{1}{2}}$, $x^{\frac{3}{2}}$, $x^{\frac{5}{2}}$, $x^{\frac{1}{3}}$, $x^{\frac{2}{3}}$, &c. for $\sqrt{x}$, $\sqrt{x^3}$, $\sqrt{x^5}$, $\sqrt[3]{x}$, $\sqrt[3]{x^2}$, &c. and of $x^{-\frac{1}{2}}$, $x^{-\frac{2}{3}}$, $x^{-\frac{1}{4}}$, &c. for $\frac{1}{\sqrt{x}}$, $\frac{1}{\sqrt[3]{x^2}}$, $\frac{1}{\sqrt[4]{x}}$, &c. And this by the Rule of Analogy, as may be apprehended from ſuch Geometrical Progreſſions as theſe; x^3, $x^{\frac{5}{2}}$, x^2, $x^{\frac{3}{2}}$, x, $x^{\frac{1}{2}}$, x^0 (or 1,) $x^{-\frac{1}{2}}$, x^{-1}, $x^{-\frac{3}{2}}$, x^{-2}, &c.

8. In the ſame manner for $\frac{aa}{x}-\frac{aab}{x^2}+\frac{aab^2}{x^3}$, &c. may be wrote $a^2x^{-1}-a^2bx^{-2}+a^2b^2x^{-3}$, &c.

9. And thus inſtead of $\sqrt{aa-xx}$ may be wrote $\overline{aa-xx}|^{\frac{1}{2}}$; and $\overline{aa-xx}|^{2}$ inſtead of the Square of $aa-xx$; and $\overline{\frac{abb-y^3}{by+yy}}\Big|^{\frac{1}{3}}$ inſtead of $\sqrt[3]{\frac{ab^2-y^3}{by+yy}}$: And the like of others.

10. So that we may not improperly diſtinguiſh Powers into Affirmative and Negative, Integral and Fractional.

Examples of Reduction by Extraction of Roots.

11. The Quantity $aa+xx$ being propoſed, you may thus extract its Square-Root.

$$\begin{array}{l}
aa+xx\ \Big(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\frac{7x^{10}}{256a^9}-\frac{21x^{12}}{1024a^{11}},\ \&c.\\
\underline{aa\qquad\qquad}\\
0+xx\\
\underline{\quad +xx+\frac{x^4}{4a^2}}\\
\qquad\qquad -\frac{x^4}{4a^2}\\
\qquad\qquad \underline{-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}}\\
\qquad\qquad\qquad\quad +\frac{x^6}{8a^4}-\frac{x^8}{64a^6}\\
\qquad\qquad\qquad\quad \underline{+\frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8}+\frac{x^{12}}{256a^{10}}}\\
\qquad\qquad\qquad\qquad\qquad -\frac{5x^8}{64a^6}+\frac{x^{10}}{64a^8}-\frac{x^{12}}{256a^{10}},\ \&c.\\
\qquad\qquad\qquad\qquad\qquad \underline{-\frac{5x^8}{64a^6}-\frac{5x^{10}}{128a^8}+\frac{5x^{12}}{512a^{10}}}\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad +\frac{7x^{10}}{128a^8}-\frac{7x^{12}}{512a^{10}},\ \&c.\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{+\frac{7x^{10}}{128a^8}+\frac{7x^{12}}{256a^{10}}}\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\frac{21x^{12}}{512a^{10}},\ \&c.
\end{array}$$

So that the Root is found to be $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}$, &c. Where it may be obſerved, that towards the end of the Operation I neglect all thoſe Terms, whoſe Dimenſions would exceed the Dimenſions of the laſt Term, to which I intend only to continue the Root, ſuppoſe to $\frac{x^{12}}{a^{11}}$.

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12. Alſo the Order of the Terms may be inverted in this manner $xx+aa$, in which caſe the Root will be found to be $x+\frac{aa}{2x}-\frac{a^4}{8x^3}+\frac{a^6}{16x^5}-\frac{5a^8}{128x^7}$, &c.

13. Thus the Root of $aa-xx$ is $a-\frac{xx}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}$ &c.

14. The Root of $x-xx$ is $x^{\frac{1}{2}}-\frac{1}{2}x^{\frac{3}{2}}-\frac{1}{8}x^{\frac{5}{2}}-\frac{1}{16}x^{\frac{7}{2}}$, &c.

15. Of $aa+bx-xx$ is $a+\frac{bx}{2a}-\frac{xx}{2a}-\frac{b^2x^2}{8a^3}$, &c.

16. And $\sqrt{\frac{1+axx}{1-bxx}}$ is $\frac{1+\frac{1}{2}ax^2-\frac{1}{8}a^2x^4+\frac{1}{16}a^3x^6,\ \&c.}{1-\frac{1}{2}bx^2-\frac{1}{8}b^2x^4-\frac{1}{16}b^3x^6,\ \&c.}$ and moreover by actually dividing, it becomes

$$\begin{array}{llll} 1 & +\frac{1}{2}bx^2 & +\frac{3}{8}b^2x^4 & +\frac{5}{16}b^3x^6,\ \&c. \\ & +\frac{1}{2}a & +\frac{1}{4}ab & +\frac{3}{16}ab^2 \\ & & -\frac{1}{8}a^2 & -\frac{1}{16}a^2b \\ & & & +\frac{1}{16}a^3 \end{array}$$

17. But theſe Operations, by due preparation, may very often be abbreviated; as in the foregoing Example to find $\sqrt{\frac{1+axx}{1-bxx}}$, if the Form of the Numerator and Denominator had not been the ſame, I might have multiply'd each by $\sqrt{1-bxx}$, which would have produced $\frac{\sqrt{1+ax^2-abx^4 \atop -b}}{1-bxx}$, and the reſt of the work might have been performed by extracting the Root of the Numerator only, and then dividing by the Denominator.

18. From hence I imagine it will ſufficiently appear, by what means any other Roots may be extracted, and how any compound Quantities, however entangled with Radicals or Denominators, (ſuch as $x^3+\frac{\sqrt{x-\sqrt{1-xx}}}{\sqrt[3]{axx+x^3}}-\frac{\sqrt[5]{x^3+2x^5-x^{\frac{3}{2}}}}{\sqrt[3]{x+xx}-\sqrt{2x-x^{\frac{2}{3}}}}$) may be reduced to infinite Series conſiſting of ſimple Terms.

Of the Reduction of affected Equations.

19. As to affected Equations, we muſt be ſomething more particular in explaining how their Roots are to be reduced to ſuch Series as theſe; becauſe their Doctrine in Numbers, as hitherto deliver'd by Mathematicians, is very perplexed, and incumber'd with ſuperfluous Operations, ſo as not to afford proper Specimens for performing the Work in Species. I ſhall therefore firſt ſhew how the Reſolu-

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Reſolution of affected Equations may be compendiouſly perform'd in Numbers, and then I ſhall apply the ſame to Species.

20. Let this Equation $y^3 - 2y - 5 = 0$ be propoſed to be reſolved, and let 2 be a Number (any how found) which differs from the true Root leſs than by a tenth part of itſelf. Then I make $2 + p = y$, and ſubſtitute $2 + p$ for y in the given Equation, by which is produced a new Equation $p^3 + 6p^2 + 10p - 1 = 0$, whoſe Root is to be ſought for, that it may be added to the Quote. Thus rejecting $p^3 + 6p^2$ becauſe of its ſmallneſs, the remaining Equation $10p - 1 = 0$, or $p = 0,1$, will approach very near to the truth. Therefore I write this in the Quote, and ſuppoſe $0,1 + q = p$, and ſubſtitute this fictitious Value of p as before, which produces $q^3 + 6,3q^2 + 11,23q + 0,061 = 0$. And ſince $11,23q + 0,061 = 0$ is near the truth, or $q = -0,0054$ nearly, (that is, dividing 0,061 by 11,23, till ſo many Figures ariſe as there are places between the firſt Figures of this, and of the principal Quote excluſively, as here there are two places between 2 and 0,005) I write $-0,0054$ in the lower part of the Quote, as being negative; and ſuppoſing $-0,0054 + r = q$, I ſubſtitute this as before. And thus I continue the Operation as far as I pleaſe, in the manner of the following Diagram:

$y^3 - 2y - 5 = 0$	$+2,10000000$ $-0,00544852$ $+2,09455148$, &c. $= y$
$2 + p = y$. $\quad +y^3$ $-2y$ -5	$+8 + 12p + 6p^2 + p^3$ $-4 - 2p$ -5
The Sum	$-1 + 10p + 6p^2 + p^3$
$0,1 + q = p$. $\quad +p^3$ $+6p^2$ $+10p$ -1	$+0,001 + 0,03q + 0,3q^2 + q^3$ $+0,06 + 1,2 + 6,$ $+1, + 10,$ $-1,$
The Sum	$0,061 + 11,23q + 6,3q^2 + q^3$
$-0,0054 + r = q$. $\quad q^3$ $+6,3q^2$ $+11,23q$ $+0,061$	$-0,000000157464 + 0,00008748r - 0,0162r^2 + r^3$ $+0,000183708 - 0,06804 + 6,3$ $-0,060642 + 11,23$ $+0,061$
The Sum	$+0,0005416 + 11,162r$
$-0,00004852 + s = r$.	

21. But the Work may be much abbreviated towards the end by this Method, eſpecially in Equations of many Dimenſions. Having firſt determin'd how far you intend to extract the Root, count ſo many places after the firſt Figure of the Coefficient of the laſt Term but one, of the Equations that reſult on the right ſide of the Diagram, as there remain places to be fill'd up in the Quote, and reject the Decimals that follow. But in the laſt Term the Decimals may be neglected, after ſo many more places as are the decimal places that are fill'd up in the Quote. And in the antepenultimate Term reject all that are after ſo many fewer places. And ſo on, by proceeding Arithmetically, according to that Interval of places: Or, which is the ſame thing, you may cut off every where ſo many Figures as in the penultimate Term, ſo that their loweſt places may be in Arithmetical Progreſſion, according to the Series of the Terms, or are to be ſuppos'd to be ſupply'd with Cyphers, when it happens otherwiſe. Thus in the preſent Example, if I deſired to continue the Quote no farther than to the eighth place of Decimals, when I ſubſtituted $0,0054 + r$ for q, where four decimal places are compleated in the Quote, and as many remain to be compleated, I might have omitted the Figures in the five inferior places, which therefore I have mark'd or cancell'd by little Lines drawn through them; and indeed I might alſo have omitted the firſt Term, r^3, although its Coefficient be 0,99999. Thoſe Figures therefore being expunged, for the following Operation there ariſes the Sum $0,0005416 + 11,162r$, which by Diviſion, continued as far as the Term preſcribed, gives $-0,00004852$ for r, which compleats the Quote to the Period required. Then ſubtracting the negative part of the Quote from the affirmative part, there ariſes 2,09455148 for the Root of the propoſed Equation.

22. It may likewiſe be obſerved, that at the beginning of the Work, if I had doubted whether $0,1 + p$ was a ſufficient Approximation to the Root, inſtead of $10p - 1 = 0$, I might have ſuppos'd that $6p^2 + 10p - 1 = 0$, and ſo have wrote the firſt Figure of its Root in the Quote, as being nearer to nothing. And in this manner it may be convenient to find the ſecond, or even the third Figure of the Quote, when in the ſecondary Equation, about which you are converſant, the Square of the Coefficient of the penultimate Term is not ten times greater than the Product of the laſt Term multiply'd into the Coefficient of the antepenultimate Term. And indeed you will often ſave ſome pains, eſpecially in Equations of many Dimenſions, if you ſeek for all the Figures

to be added to the Quote in this manner; that is, if you extract the leſſer Root out of the three laſt Terms of its ſecondary Equation: For thus you will obtain, at every time, as many Figures again in the Quote.

23. And now from the Reſolution of numeral Equations, I ſhall proceed to explain the like Operations in Species; concerning which, it is neceſſary to obſerve what follows.

24. Firſt; that ſome one of the ſpecious or literal Coefficients, if there are more than one, ſhould be diſtinguiſh'd from the reſt, which either is, or may be ſuppos'd to be, much the leaſt or greateſt of all, or neareſt to a given Quantity. The reaſon of which is, that becauſe of its Dimenſions continually increaſing in the Numerators, or the Denominators of the Terms of the Quote, thoſe Terms may grow leſs and leſs, and therefore the Quote may conſtantly approach to the Root required; as may appear from what is ſaid before of the Species x, in the Examples of Reduction by Diviſion and Extraction of Roots. And for this Species, in what follows, I ſhall generally make uſe of x or z; as alſo I ſhall uſe y, p, q, r, s, &c. for the Radical Species to be extracted.

25. Secondly, when any complex Fractions, or ſurd Quantities, happen to occur in the propoſed Equation, or to ariſe afterwards in the Proceſs, they ought to be removed by ſuch Methods as are ſufficiently known to Analyſts. As if we ſhould have $y^3 + \frac{bb}{b-x}y^2 - x^3 = 0$, multiply by $b - x$, and from the Product $by^3 - xy^3 + b^2y^2 - bx^3 + x^4 = 0$ extract the Root y. Or we might ſuppoſe $y \times \overline{b-x} = v$, and then writing $\frac{v}{b-x}$ for y, we ſhould have $v^3 + b^2v^2 - b^3x^3 + 3b^2x^4 - 3bx^5 + x^6 = 0$, whence extracting the Root v, we might divide the Quote by $b - x$, in order to obtain y. Alſo if the Equation $y^3 - xy^{\frac{5}{2}} + x^{\frac{4}{3}} = 0$ were propoſed, we might put $y^{\frac{1}{2}} = v$, and $x^{\frac{1}{3}} = z$, and ſo writing vv for y, and z^3 for x, there will ariſe $v^6 - z^3v + z^4 = 0$; which Equation being reſolved, y and x may be reſtored. For the Root will be found $v = z + z^3 + 6z^5$, &c. and reſtoring y and x, we have $y^{\frac{1}{2}} = x^{\frac{1}{3}} + x + 6x^{\frac{5}{3}}$, &c. then ſquaring, $y = x^{\frac{2}{3}} + 2x^{\frac{4}{3}} + 13x^2$, &c.

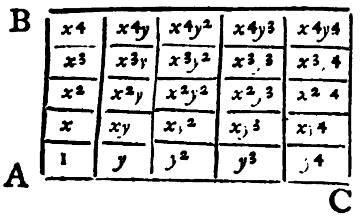
26. After the ſame manner if there ſhould be found negative Dimenſions of x and y, they may be removed by multiplying by the ſame x and y. As if we had the Equation $x^3 + 3x^2y^{-1} - 2x^{-1} - 16y^{-3} = 0$, multiply by x and y^3, and there would ariſe $x^4y^3 + 3x^3y^2 - 2y^3 - 16x = 0$. And if the Equation were $x = \frac{aa}{y} - \frac{2a^3}{y^2} + \frac{3a^4}{y^3}$,

by multiplying into y^3 there would ariſe $xy^3 = a^2y^2 - 2a^3y + 3a^4$. And ſo of others.

27. Thirdly, when the Equation is thus prepared, the work begins by finding the firſt Term of the Quote; concerning which, as alſo for finding the following Terms, we have this general Rule, when the indefinite Species (x or z) is ſuppoſed to be ſmall; to which Caſe the other two Caſes are reducible.

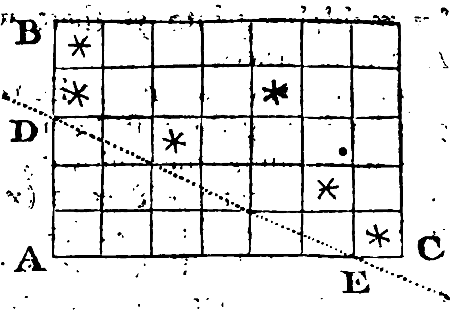
28. Of all the Terms, in which the Radical Species ($y, p, q,$ or r, &c.) is not found, chuſe the loweſt in reſpect of the Dimenſions of the indefinite Species (x or z, &c.) then chuſe another Term in which that Radical Species is found, ſuch as that the Progreſſion of the Dimenſions of each of the fore-mentioned Species, being continued from the Term firſt aſſumed to this Term, may deſcend as much as may be, or aſcend as little as may be. And if there are any other Terms, whoſe Dimenſions may fall in with this Progreſſion continued at pleaſure, they muſt be taken in likewiſe. Laſtly, from theſe Terms thus ſelected, and made equal to nothing, find the Value of the ſaid Radical Species, and write it in the Quote.

29. But that this Rule may be more clearly apprehended, I ſhall explain it farther by help of the following Diagram. Making a right Angle BAC, divide its ſides AB, AC, into equal parts, and raiſing Perpendiculars, diſtribute the Angular Space into equal Squares or Parallelograms, which you may conceive to be denominated from the Dimenſions of the Species x and y, as they are here inſcribed. Then, when any Equation is propoſed, mark ſuch of the Parallelograms as correſpond to all its Terms, and let a Ruler be apply'd to two, or perhaps more, of the Parallelograms ſo mark'd, of which let one be the loweſt in the left-hand Column at AB, the other touching the Ruler towards the right-hand; and let all the reſt, not touching the Ruler, lie above it. Then ſelect thoſe Terms of the Equation which are repreſented by the Parallelograms that touch the Ruler, and from them find the Quantity to be put in the Quote.



x^4	x^4y	x^4y^2	x^4y^3	x^4y^4
x^3	x^3y	x^3y^2	x^3y^3	x^3y^4
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4
x	xy	xy^2	xy^3	xy^4
1	y	y^2	y^3	y^4

30. Thus to extract the Root y out of the Equation $y^6 - 5xy^5 + \frac{x^3}{a}y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$, I mark the Parallelograms belonging



ing to the Terms of this Equation with the Mark *, as you ſee here done. Then I apply the Ruler DE to the lower of the Parallelograms mark'd in the left-hand Column, and I make it turn round towards the right-hand from the lower to the upper, till it begins in like manner to touch another, or perhaps more, of the Parallelograms that are mark'd; and I ſee that the places ſo touch'd belong to x^3, x^2y^2, and y^6. Therefore from the Terms $y^6-7a^2x^2y^2+6a^3x^3$, as if equal to nothing, (and moreover, if you pleaſe, reduced to $v^6-7v^2+6=0$, by making $y=v\sqrt{ax}$,) I ſeek the Value of y, and find it to be four-fold, $+\sqrt{ax}$, $-\sqrt{ax}$, $+\sqrt{2ax}$, and $-\sqrt{2ax}$, of which I may take any one for the initial Term of the Quote, according as I deſign to extract this or that Root of the given Equation.

31. Thus having the Equation $y^3-by^2+9bx^2-x^3=0$, I chuſe the Terms $-by^2+9bx^2$, and thence I obtain $+3x$ for the initial Term of the Quote.

32. And having $y^3+axy+aay-x^3-2a^3=0$, I make choice of $y^3+a^2y-2a^3$, and its Root $+a$ I write in the Quote.

33. Alſo having $x^2y^5-3c^4xy^2-c^5x^2+c^7=0$, I ſelect $x^2y^5+c^7$, which gives $-\sqrt[5]{\frac{c^7}{x^2}}$ for the firſt Term of the Quote. And the like of others.

34. But when this Term is found, if its Power ſhould happen to be negative, I depreſs the Equation by the ſame Power of the indefinite Species, that there may be no need of depreſſing it in the Reſolution; and beſides, that the Rule hereafter deliver'd, for the ſuppreſſion of ſuperfluous Terms, may be conveniently apply'd. Thus the Equation $8z^6y^3+az^6y^2-27a^9=0$ being propoſed, whoſe Root is to begin by the Term $\frac{3a^3}{2z^2}$, I depreſs by z^2, that it may become $8z^4y^3+az^4y^2-27a^9z^{-2}=0$, before I attempt the Reſolution.

35. The ſubſequent Terms of the Quotes are derived by the ſame Method, in the Progreſs of the Work, from their ſeveral ſecondary Equations, but commonly with leſs trouble. For the whole affair is perform'd by dividing the loweſt of the Terms affected with the indefinitely ſmall Species, (x, x^2, x^3, &c.) without the Radical Species, (p, q, r, &c.) by the Quantity with which that radical Species

of one Dimenſion only is affected, without the other indefinite Species, and by writing the Reſult in the Quote. So in the following Example, the Terms $-\frac{x}{4}$, $\frac{xx}{64a}$, $\frac{131x^3}{512a^2}$, &c. are produced by dividing $-a^2x$, $\frac{1}{16}ax^2$, $\frac{131}{128}x^3$, &c. by $4aa$.

36. Theſe things being premiſed, it remains now to exhibit the Praxis of Reſolution. Therefore, let the Equation $y^3+a^2y+axy-2a^3-x^3=0$ be propoſed to be reſolved. And from its Terms $y^3+a^2y-2a^3=0$, being a fictitious Equation, by the third of the foregoing Premiſes, I obtain $y-a=0$, and therefore I write $+a$ in the Quote. Then becauſe $+a$ is not the compleat Value of y, I put $a+p=y$, and inſtead of y, in the Terms of the Equation written in the Margin, I ſubſtitute $a+p$, and the Terms reſulting (p^3+3ap^2+axp, &c.) I again write in the Margin; from which again, according to the third of the Premiſes, I ſelect the Terms $+4a^2p+a^2x=0$ for a fictitious Equation, which giving $p=-\frac{1}{4}x$, I write $-\frac{1}{4}x$ in the Quote. Then becauſe $-\frac{1}{4}x$ is not the accurate Value of p, I put $-\frac{1}{4}x+q=p$, and in the marginal Terms for p I ſubſtitute $-\frac{1}{4}x+q$, and the reſulting Terms ($q^3-\frac{3}{4}xq^2+3aq^2$, &c.) I again write in the Margin, out of which, according to the foregoing Rule, I again ſelect the Terms $4a^2q-\frac{1}{16}ax^2=0$ for a fictitious Equation, which giving $q=\frac{xx}{64a}$, I write $\frac{xx}{64a}$ in the Quote. Again, ſince $\frac{xx}{64a}$ is not the accurate Value of q, I make $\frac{xx}{64a}+r=q$, and inſtead of q I ſubſtitute $\frac{xx}{64a}+r$ in the marginal Terms. And thus I continue the Proceſs at pleaſure, as the following Diagram exhibits to view.

$y^3+a^2y-2a^3+axy-x^3=0.$		$y=a-\frac{x}{4}+\frac{x^2}{64a}+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$ &c.
$+a+p=y.$	$+y^3$ $+axy$ $+a^2y$ $-x^3$ $-2a^3$	$+a^3\ +3a^2p+3ap^2+p^3$ $+a^2x+axp$ $+a^3\ +a^2p$ $-x^3$ $-2a^3$
$-\frac{1}{4}x+q=p.$	$+p^3$ $+3ap^2$ $+axp$ $+4a^2p$ $+a^2x$ $-x^3$	$-\frac{1}{64}x^3\ +\frac{3}{16}x^2q-\frac{3}{4}xq^2+q^3$ $+\frac{3}{16}ax^2-\frac{3}{2}axq\ +3aq^2$ $-\frac{1}{4}ax^2+axq$ $-a^2x\ +4a^2q$ $+a^2x$ $-x^3$
$+\frac{x^2}{64a}+r=q.$	$+q^3$ $-\frac{3}{4}xq^2$ $+3aq^2$ $+\frac{3}{16}x^2q$ $-\frac{1}{2}axq$ $+4a^2q$ $-\frac{65}{64}x^3$ $-\frac{1}{16}ax^2$	* * $+\frac{3x^4}{4096a}$ * $+\frac{3}{32}x^2r+3ar^2$ $+\frac{3x^4}{1024a}$ * $+\frac{3}{16}x^2r$ $-\frac{1}{128}x^3\ -\frac{1}{2}axr$ $+\frac{1}{16}ax^2\ +4a^2r$ $-\frac{65}{64}x^3$ $-\frac{1}{16}ax^2$
$+4a^2-\frac{1}{2}ax+\frac{9}{32}x^2)+\frac{131}{128}x^3-\frac{15x^4}{4096a}\ (+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$		

37. If it were required to continue the Quote only to a certain Period, that x, for inſtance, in the laſt Term ſhould not aſcend beyond a given Dimenſion; as I ſubſtitute the Terms, I omit ſuch as I foreſee will be of no uſe. For which this is the Rule, that after the firſt Term reſulting in the collateral Margin from every Quantity, ſo many Terms are to be added to the right-hand, as the Index of the higheſt Power required in the Quote exceeds the Index of that firſt reſulting Term.

38. As in the preſent Example, if I deſired that the Quote, (or the Species x in the Quote,) ſhould aſcend no higher than to four Dimenſions, I omit all the Terms after x^4, and put only one after x^3. Therefore

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Therefore the Terms after the Mark * are to be conceived to be expunged. And thus the Work being continued till at laſt we come to the Terms $\frac{15x^4}{4096a}-\frac{131x^3}{128}+4a^2r-\frac{1}{2}axr$, in which p, q, r, or s, &c. repreſenting the Supplement of the Root to be extracted, are only of one Dimenſion; we may find ſo many Terms by Diviſion, $\left(+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}\right)$ as we ſhall ſee wanting to compleat the Quote. So that at laſt we ſhall have $y=a-\frac{1}{4}x+\frac{xx}{64a}+\frac{131x^3}{512a^2}+\frac{509x^4}{16384a^3}$ &c.

39. For the ſake of farther Illuſtration, I ſhall propoſe another Example to be reſolved. From the Equation $\frac{1}{5}y^5-\frac{1}{4}y^4+\frac{1}{3}y^3-\frac{1}{2}y^2+y-z=0$, let the Quote be found only to the fifth Dimenſion, and the ſuperfluous Terms be rejected after the Mark, &c.

$\frac{1}{5}y^5-\frac{1}{4}y^4+\frac{1}{3}y^3-\frac{1}{2}y^2+y-z=0.$		$y=z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4+\frac{1}{120}z^5$, &c.
$z+p=y.$	$+\frac{1}{5}y^5$	$+\frac{1}{5}z^5$, &c.
	$-\frac{1}{4}y^4$	$-\frac{1}{4}z^4-z^3p$, &c.
	$+\frac{1}{3}y^3$	$+\frac{1}{3}z^3+z^2p+zp^2$, &c.
	$-\frac{1}{2}y^2$	$-\frac{1}{2}z^2-zp-\frac{1}{2}p^2$
	$+y$	$+z\quad +p$
	$-z$	$-z$
$\frac{1}{2}z^2+q=p.$	$+zp^2$	$+\frac{1}{4}z^5$, &c.
	$-\frac{1}{2}p^2$	$-\frac{1}{8}z^4-\frac{1}{2}z^2q$, &c.
	$-z^3p$	$-\frac{1}{2}z^5$, &c.
	$+z^2p$	$+\frac{1}{2}z^4+z^2q$
	$-zp$	$-\frac{1}{2}z^3-zq$
	$+p$	$+\frac{1}{2}z^2+q$
	$+\frac{1}{5}z^5$	$+\frac{1}{5}z^5$
	$-\frac{1}{4}z^4$	$-\frac{1}{4}z^4$
	$+\frac{1}{3}z^3$	$+\frac{1}{3}z^3$
	$-\frac{1}{2}z^2$	$-\frac{1}{2}z^2$
$1-z+\frac{1}{2}z^2)\ \frac{1}{6}z^3-\frac{1}{8}z^4+\frac{1}{20}z^5\ (\frac{1}{6}z^3+\frac{1}{24}z^4+\frac{1}{120}z^5$		

40. And thus if we propoſe the Equation $\frac{63}{2816}y^{11}+\frac{35}{1152}y^9+\frac{5}{112}y^7+\frac{3}{40}y^5+\frac{1}{6}y^3+y-z=0$, to be reſolved only to the ninth Dimenſion of the Quote; before the Work begins we may reject the Term $\frac{63}{2816}y^{11}$; then as we operate we may reject all the Terms beyond z^9, beyond z^7 we may admit but one, and two only after

z^9; becauſe we may obſerve, that the Quote ought always to aſcend by the Interval of two Units, in this manner, z, z^3, z^5, &c. Then at laſt we ſhall have $y = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$, &c.

41. And hence an Artifice is diſcover'd, by which Equations, tho' affected *in infinitum*, and conſiſting of an infinite number of Terms, may however be reſolved. And that is, before the Work begins all the Terms are to be rejected, in which the Dimenſion of the indefinitely ſmall Species, not affected by the radical Species, exceeds the greateſt Dimenſion required in the Quote; or from which, by ſubſtituting inſtead of the radical Species, the firſt Term of the Quote found by the Parallelogram as before, none but ſuch exceeding Terms can ariſe. Thus in the laſt Example I ſhould have omitted all the Terms beyond y^9, though they went on *ad infinitum*. And ſo in this Equation

$$0 = \begin{cases} -8 + z^2 - 4z^4 + 9z^6 - 16z^8, \text{ \&c.} \\ +y \text{ in } z^2 - 2z^4 + 3z^6 - 4z^8, \text{ \&c.} \\ -y^2 \text{ in } z^2 - z^4 + z^6 - z^8, \text{ \&c.} \\ +y^3 \text{ in } z^2 - \frac{1}{2}z^4 + \frac{1}{3}z^6 - \frac{1}{4}z^8, \text{ \&c.} \end{cases}$$

that the Cubick Root may be extracted only to four Dimenſions of z, I omit all the Terms *in infinitum* beyond $+y^3$ in $z^2 - \frac{1}{2}z^4 + \frac{1}{3}z^6$, and all beyond $-y^2$ in $z^2 - z^4 + z^6$, and all beyond $+y$ in $z^2 - 2z^4$, and beyond $-8 + z^2 - 4z^4$. And therefore I aſſume this Equation only to be reſolved, $\frac{1}{3}z^6y^3 - \frac{1}{2}z^4y^3 + z^2y^3 - z^6y^2 + z^4y^2 - z^2y^2 - 2z^4y + z^2y - 4z^4 + z^2 - 8 = 0$. Becauſe $2z^{-\frac{2}{3}}$, (the firſt Term of the Quote,) being ſubſtituted inſtead of y in the reſt of the Equation depreſs'd by $z^{\frac{2}{3}}$, gives every where more than four Dimenſions.

42. What I have ſaid of higher Equations may alſo be apply'd to Quadraticks. As if I deſired the Root of this Equation

$$0 = \begin{cases} y^2 \\ -y \text{ in } a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} \text{ \&c.} \\ + \frac{x^4}{4a^2} \end{cases}$$

as far as the Period x^6, I omit all the Terms *in infinitum*, beyond $-y$ in $a + x + \frac{x^2}{a}$, and aſſume only this Equation, $y^2 - ay - xy - \frac{x^2}{a}y + \frac{x^4}{4a^2} = 0$. This I reſolve either in the uſual manner, by making

$=\frac{1}{2}a+\frac{1}{2}x+\frac{x^2}{2a}-\sqrt{\frac{1}{4}a^2+\frac{1}{2}ax+\frac{3}{4}x^2+\frac{x^3}{2a}}$; or more expeditiouſly by the Method of affected Equations deliver'd before, by which we ſhall have $y=\frac{x^4}{4a^3}-\frac{x^5}{4a^4}$ *, where the laſt Term required vaniſhes, or becomes equal to nothing.

43. Now after that Roots are extracted to a convenient Period, they may ſometimes be continued at pleaſure, only by obſerving the Analogy of the Series. So you may for ever continue this $z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4+\frac{1}{120}z^5$, &c. (which is the Root of the infinite Equation $z=y+\frac{1}{2}y^2+\frac{1}{3}y^3+\frac{1}{4}y^4$, &c.) by dividing the laſt Term by theſe Numbers in order 2, 3, 4, 5, 6, *&c.* And this, $z-\frac{1}{6}z^3+\frac{1}{120}z^5-\frac{1}{5040}z^7+\frac{1}{362880}z^9$, &c. may be continued by dividing by theſe Numbers 2×3, 4×5, 6×7, 8×9, *&c.* Again, the Series $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}$, &c. may be continued at pleaſure, by multiplying the Terms reſpectively by theſe Fractions, $\frac{1}{2}$, $-\frac{1}{4}$, $-\frac{3}{6}$, $-\frac{5}{8}$, $-\frac{7}{10}$, *&c.* And ſo of others.

44. But in diſcovering the firſt Term of the Quote, and ſometimes of the ſecond or third, there may ſtill remain a difficulty to be overcome. For its Value, ſought for as before, may happen to be ſurd, or the inextricable Root of an high affected Equation. Which when it happens, provided it be not alſo impoſſible, you may repreſent it by ſome Letter, and then proceed as if it were known. As in the Example $y^3+axy+a^2y-x^3-2a^3=0$: If the Root of this Equation $y^3+a^2y-2a^3=0$, had been ſurd, or unknown, I ſhould have put any Letter b for it, and then have perform'd the Reſolution as follows, ſuppoſe the Quote found only to the third Dimenſion.

$$y^3+aay+axy-2a^3-x^3=0. \quad \text{Make } a^2+3b^2=c^2, \text{ then}$$

$$y=b-\frac{abx}{c^2}+\frac{a^4bx^2}{c^6}+\frac{x^3}{c^2}+\frac{a^3b^3x^3}{c^8}-\frac{a^5bx^3}{c^8}+\frac{a^5b^3x^3}{c^{10}} \text{ \&c.}$$

$b+p=y.$	$+y^3$	$+b^3 \quad +3b^2p+3bp^2+p^3.$
	$+axy$	$+abx+axp$
	$+a^2y$	$+a^2b+a^2p$
	$-x^3$	$-x^3$
	$-2a^3$	$-2a^3$
$-\frac{abx}{c^2}+q=p.$	$+p^3$	$-\frac{a^3b^3x^3}{c^6}$ &c.
	$+3bp^2$	$+\frac{3a^2b^3x^2}{c^4}-\frac{6ab^2x}{c^2}q$ &c.
	$+axp$	$-\frac{a^2bx^2}{c^2} \quad +axq$
	$+c^2p$	$-abx \quad +c^2q$
	$-x^3$	$-x^3$
	$+abx$	$+abx$

$$c^2+ax-\frac{6ab^2x}{c^2}\Big)\ \frac{a^4bx^2}{c^4}+x^3+\frac{a^3b^3x^3}{c^6}\ \Big(\frac{a^4bx^2}{c^6}+\frac{x^3}{c^2}+\frac{a^3b^3x^3}{c^8} \text{ \&c.}$$

45. Here writing b in the Quote, I ſuppoſe $b+p=y$, and then for y I ſubſtitute as you ſee. Whence proceeds p^3+3bp^2, &c. rejecting the Terms $b^3+a^2b-2a^3$, as being equal to nothing: For b is ſuppos'd to be a Root of this Equation $y^3+a^2y-2a^3=0$. Then the Terms $3b^2p+a^2p+abx$ give $\frac{-abx}{3b^2+a^2}$ to be ſet in the Quote, and $\frac{-abx}{a^2+3b^2}+q$ to be ſubſtituted for p.

46. But for brevity's ſake I write cc for $aa+3bb$, yet with this caution, that $aa+3bb$ may be reſtored, whenever I perceive that the Terms may be abbreviated by it. When the Work is finiſh'd, I aſſume ſome Number for a, and reſolve this Equation $y^3+a^2y-2a^3=0$, as is ſhewn above concerning Numeral Equations; and I ſubſtitute for b any one of its Roots, if it has three Roots. Or rather, I deliver ſuch Equations from Species, as far as I can, eſpecially from the indefinite Species, and that after the manner before inſinuated. And for the reſt only, if any remain that cannot be expunged, I put Numbers. Thus $y^3+a^2y-2a^3=0$ will be freed from a, by dividing the Root by a, and it will become $y^3+y-2=0$, whoſe Root being found, and multiply'd by a, muſt be ſubſtituted inſtead of b.

47.

47. Hitherto I have ſuppos'd the indefinite Species to be little. But if it be ſuppos'd to approach nearly to a given Quantity, for that indefinitely ſmall difference I put ſome Species, and that being ſubſtituted, I ſolve the Equation as before. Thus in the Equation $\frac{1}{5}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 + y + a - x = 0$, it being known or ſuppos'd that x is nearly of the ſame Quantity as a, I ſuppoſe z to be their difference; and then writing $a + z$ or $a - z$ for x, there will ariſe $\frac{1}{5}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 + y \pm z = 0$, which is to be ſolved as before.

48. But if that Species be ſuppos'd to be indefinitely great, for its Reciprocal, which will therefore be indefinitely little, I put ſome Species, which being ſubſtituted, I proceed in the Reſolution as before. Thus having $y^3 + y^2 + y - x^3 = 0$, where x is known or ſuppos'd to be very great, for the reciprocally little Quantity $\frac{1}{x}$ I put z, and ſubſtituting $\frac{1}{z}$ for x, there will ariſe $y^3 + y^2 + y - \frac{1}{x^3} = 0$, whoſe Root is $y = \frac{1}{z} - \frac{1}{3} - \frac{2}{9}z + \frac{7}{81}z^2 + \frac{5}{81}z^3$, &c. where x being reſtored, if you pleaſe, it will be $y = x - \frac{1}{3} + \frac{2}{9x} + \frac{7}{81x^2} + \frac{5}{81x^3}$, &c.

49. If it ſhould happen that none of theſe Expedients ſhould ſucceed to your deſire, you may have recourſe to another. Thus in the Equation $y^4 - x^2y^2 + xy^2 + 2y^2 - 2y + 1 = 0$, whereas the firſt Term ought to be obtain'd from the Suppoſition that $y^4 + 2y^2 - 2y + 1 = 0$, which yet admits of no poſſible Root; you may try what can be done another way. As you may ſuppoſe that x is but little different from $+2$, or that $2 + z = x$. Then ſubſtituting $2 + z$ inſtead of x, there will ariſe $y^4 - z^2y^2 - 3zy^2 - 2y + 1 = 0$, and the Quote will begin from $+1$. Or if you ſuppoſe x to be indefinitely great, or $\frac{1}{x} = z$, you will have $y^4 - \frac{y^2}{z^2} + \frac{y^2}{z} + 2y^2 - 2y + 1 = 0$, and $+z$ for the initial Term of the Quote.

50. And thus by proceeding according to ſeveral Suppoſitions, you may extract and expreſs Roots after various ways.

51. If you ſhould deſire to find after how many ways this may be done, you muſt try what Quantities, when ſubſtituted for the indefinite Species in the propoſed Equation, will make it diviſible by y, $+$ or $-$ ſome Quantity, or by y alone. Which, for Example ſake, will happen in the Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$,

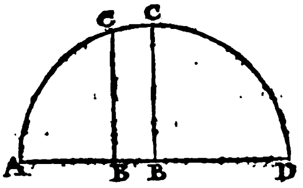
by ſubſtituting $+a$, or $-a$, or $-2a$, or $\overline{-2a^3}|^{\frac{1}{3}}$, &c. inſtead of x. And thus you may conveniently ſuppoſe the Quantity x to differ little from $+a$, or $-a$, or $-2a$, or $\overline{-2a^3}|^{\frac{1}{3}}$, and thence you may extract the Root of the Equation propoſed after ſo many ways. And perhaps alſo after ſo many other ways, by ſuppoſing thoſe differences to be indefinitely great. Beſides, if you take for the indefinite Quantity this or that of the Species which expreſs the Root, you may perhaps obtain your deſire after other ways. And farther ſtill, by ſubſtituting any fictitious Values for the indefinite Species, ſuch as $az+bz^2$, $\frac{a}{b+z}$, $\frac{a+cz}{b+z}$, &c. and then proceeding as before in the Equations that will reſult.

52. But now that the truth of theſe Concluſions may be manifeſt; that is, that the Quotes thus extracted, and produced *ad libitum*, approach ſo near to the Root of the Equation, as at laſt to differ from it by leſs than any aſſignable Quantity, and therefore when infinitely continued, do not at all differ from it: You are to conſider, that the Quantities in the left-hand Column of the right-hand ſide of the Diagrams, are the laſt Terms of the Equations whoſe Roots are p, q, r, s, &c. and that as they vaniſh, the Roots p, q, r, s, &c. that is, the differences between the Quote and the Root ſought, vaniſh at the ſame time. So that the Quote will not then differ from the true Root. Wherefore at the beginning of the Work, if you ſee that the Terms in the ſaid Column will all deſtroy one another, you may conclude, that the Quote ſo far extracted is the perfect Root of the Equation. But if it be otherwiſe, you will ſee however, that the Terms in which the indefinitely ſmall Species is of few Dimenſions, that is, the greateſt Terms, are continually taken out of that Column, and that at laſt none will remain there, unleſs ſuch as are leſs than any given Quantity, and therefore not greater than nothing when the Work is continued *ad infinitum*. So that the Quote, when infinitely extracted, will at laſt be the true Root.

53. Laſtly, altho' the Species, which for the ſake of perſpicuity I have hitherto ſuppos'd to be indefinitely little, ſhould however be ſuppos'd to be as great as you pleaſe, yet the Quotes will ſtill be true, though they may not converge ſo faſt to the true Root. This is manifeſt from the Analogy of the thing. But here the Limits of the Roots, or the greateſt and leaſt Quantities, come to be conſider'd. For theſe Properties are in common both to finite and infinite Equations. The Root in theſe is then greateſt or leaſt,

when there is the greateſt or leaſt difference between the Sums of the affirmative Terms, and of the negative Terms; and is limited when the indefinite Quantity, (which therefore not improperly I ſuppos'd to be ſmall,) cannot be taken greater, but that the Magnitude of the Root will immediately become infinite, that is, will become impoſſible.

54. To illuſtrate this, let ACD be a Semicircle deſcribed on the Diameter AD, and BC be an Ordinate. Make AB $= x$, BC $= y$, AD $= a$. Then $y = \sqrt{ax - xx} = \sqrt{ax} - \frac{x}{2a}\sqrt{ax} - \frac{x^2}{8a^2}\sqrt{ax} - \frac{x^3}{16a^3}\sqrt{ax}$, &c. as before. Therefore BC, or y, then becomes greateſt when $\sqrt{ax}$ moſt exceeds all the Terms $\frac{x}{2a}\sqrt{ax} + \frac{x^2}{8a^2}\sqrt{ax} + \frac{x^3}{16a^3}\sqrt{ax}$, &c. that is, when $x = \frac{1}{2}a$; but it will be terminated when $x = a$. For if we take x greater than a, the Sum of all the Terms $-\frac{x}{2a}\sqrt{ax} - \frac{x^2}{8a^2}\sqrt{ax} - \frac{x^3}{16a^3}\sqrt{ax}$, &c. will be infinite. There is another Limit alſo, when $x = 0$, by reaſon of the impoſſibility of the Radical $\sqrt{-ax}$; to which Terms or Limits, the Limits of the Semicircle A, B, and D, are correſpondent.



Tranſition to the METHOD OF FLUXIONS.

55. And thus much for the Methods of Computation, of which I ſhall make frequent uſe in what follows. Now it remains, that for an Illuſtration of the Analytick Art, I ſhould give ſome Specimens of Problems, eſpecially ſuch as the nature of Curves will ſupply. But firſt it may be obſerved, that all the difficulties of theſe may be reduced to theſe two Problems only, which I ſhall propoſe concerning a Space deſcribed by local Motion, any how accelerated or retarded.

56. I. *The Length of the Space deſcribed being continually (that is, at all Times) given; to find the Velocity of the Motion at any Time propoſed.*

57. II. *The Velocity of the Motion being continually given; to find the Length of the Space deſcribed at any Time propoſed.*

58. Thus in the Equation $xx = y$, if y repreſents the Length of the Space at any time deſcribed, which (time) another Space x, by increaſing with an uniform Celerity $\dot{x}$, meaſures and exhibits as

deſcribed: Then $2\dot{x}x$ will repreſent the Celerity by which the Space y, at the ſame moment of Time, proceeds to be deſcribed; and contrary-wiſe. And hence it is, that in what follows, I conſider Quantities as if they were generated by continual Increaſe, after the manner of a Space, which a Body or Thing in Motion deſcribes.

59. But whereas we need not conſider the Time here, any farther than as it is expounded and meaſured by an equable local Motion; and beſides, whereas only Quantities of the ſame kind can be compared together, and alſo their Velocities of Increaſe and Decreaſe: Therefore in what follows I ſhall have no regard to Time formally conſider'd, but I ſhall ſuppoſe ſome one of the Quantities propoſed, being of the ſame kind, to be increaſed by an equable Fluxion, to which the reſt may be referr'd, as it were to Time; and therefore, by way of Analogy, it may not improperly receive the name of Time. Whenever therefore the word *Time* occurs in what follows, (which for the ſake of perſpicuity and diſtinction I have ſometimes uſed,) by that Word I would not have it underſtood as if I meant Time in its formal Acceptation, but only that other Quantity, by the equable Increaſe or Fluxion whereof, Time is expounded and meaſured.

60. Now thoſe Quantities which I conſider as gradually and indefinitely increaſing, I ſhall hereafter call *Fluents*, or *Flowing Quantities*, and ſhall repreſent them by the final Letters of the Alphabet v, x, y, and z; that I may diſtinguiſh them from other Quantities, which in Equations are to be conſider'd as known and determinate, and which therefore are repreſented by the initial Letters a, b, c, &c. And the Velocities by which every Fluent is increaſed by its generating Motion, (which I may call *Fluxions*, or ſimply Velocities or Celerities,) I ſhall repreſent by the ſame Letters pointed thus $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$. That is, for the Celerity of the Quantity v I ſhall put $\dot{v}$, and ſo for the Celerities of the other Quantities x, y, and z, I ſhall put $\dot{x}$, $\dot{y}$, and $\dot{z}$ reſpectively.

61. Theſe things being premiſed, I ſhall now forthwith proceed to the matter in hand; and firſt I ſhall give the Solution of the two Problems juſt now propoſed.

PROB.

PROB. I.

The Relation of the Flowing Quantities to one another being given, to determine the Relation of their Fluxions.

SOLUTION.

1. Diſpoſe the Equation, by which the given Relation is expreſs'd, according to the Dimenſions of ſome one of its flowing Quantities, ſuppoſe x, and multiply its Terms by any Arithmetical Progreſſion, and then by $\frac{\dot{x}}{x}$. And perform this Operation ſeparately for every one of the flowing Quantities. Then make the Sum of all the Products equal to nothing, and you will have the Equation required.

2. EXAMPLE I. If the Relation of the flowing Quantities x and y be $x^3 - ax^2 + axy - y^3 = 0$; firſt diſpoſe the Terms according to x, and then according to y, and multiply them in the following manner.

$$\begin{array}{l|l}
\text{Mult.}\ \ x^3 \quad - ax^2 \quad + axy - y^3 & -y^3 \quad + axy \quad \begin{array}{l}-ax^2\\+x^3\end{array} \\
\text{by}\quad \frac{3\dot{x}}{x} \ .\ \frac{2\dot{x}}{x} \ .\ \frac{\dot{x}}{x} \ .\ 0 & \frac{3\dot{y}}{y} \ .\ \frac{\dot{y}}{y} \ .\ 0 \\
\hline
\text{makes}\ \ 3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y \quad * & -3\dot{y}y^2 + a\dot{y}x \quad *
\end{array}$$

The Sum of the Products is $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$, which Equation gives the Relation between the Fluxions $\dot{x}$ and $\dot{y}$. For if you take x at pleaſure, the Equation $x^3 - ax^2 + axy - y^3 = 0$ will give y. Which being determined, it will be $\dot{x} : \dot{y} :: 3y^2 - ax : 3x^2 - 2ax + ay$.

3. Ex. 2. If the Relation of the Quantities x, y, and z, be expreſs'd by the Equation $2y^3 + x^2y - 2cyz + 3yz^2 - z^3 = 0$,

$$\begin{array}{l|l|l}
\text{Mult.}\ 2y^3 + xx \times y - z^3 & yx^2 + 2y^3 & -z^3 + 3yz^2 - 2cyz + x^2y \\
\qquad\qquad - 2cz & \qquad\quad - 2cyz & \qquad\qquad\qquad\qquad\qquad + 2y^3 \\
\qquad\qquad + 3z^2 & \qquad\quad + 3yz^2 & \\
 & \qquad\quad - z^3 & \\
\text{by}\ \ \frac{2\dot{y}}{y} \ .\ 0 \ .\ -\frac{\dot{y}}{y} & \frac{2\dot{x}}{x} \ .\ 0 . & \frac{3\dot{z}}{z} \ .\ \frac{2\dot{z}}{z} \ .\ \frac{\dot{z}}{z} \ .\ 0 \\
\hline
\text{makes}\ 4\dot{y}y^2 \quad * \quad + \frac{\dot{y}z^3}{y} & 2\dot{x}xy \quad * & -3\dot{z}z^2 + 6\dot{z}zy - 2c\dot{z}y \quad *
\end{array}$$

Wherefore the Relation of the Celerities of Flowing, or of the Fluxions $\dot{x}$, $\dot{y}$, and $\dot{z}$, is $4\dot{y}y^2 + \frac{\dot{y}x^4}{y} + 2\dot{x}xy - 3\dot{z}z^2 + 6\dot{z}zy - 2c\dot{z}y = 0$.

4. But ſince there are here three flowing Quantities, x, y, and z, another Equation ought alſo to be given, by which the Relation among them, as alſo among their Fluxions, may be intirely determined. As if it were ſuppoſed that $x + y - z = 0$. From whence another Relation among the Fluxions $\dot{x} + \dot{y} - \dot{z} = 0$ would be found by this Rule. Now compare theſe with the foregoing Equations, by expunging any one of the three Quantities, and alſo any one of the Fluxions, and then you will obtain an Equation which will intirely determine the Relation of the reſt.

5. In the Equation propos'd, whenever there are complex Fractions, or ſurd Quantities, I put ſo many Letters for each, and ſuppoſing them to repreſent flowing Quantities, I work as before. Afterwards I ſuppreſs and exterminate the aſſumed Letters, as you ſee done here.

6. Ex. 3. If the Relation of the Quantities x and y be $yy - aa - x\sqrt{aa - xx} = 0$; for $x\sqrt{aa - xx}$ I write z, and thence I have the two Equations $yy - aa - z = 0$, and $a^2x^2 - x^4 - z^2 = 0$, of which the firſt will give $2\dot{y}y - \dot{z} = 0$, as before, for the Relation of the Celerities $\dot{y}$ and $\dot{z}$, and the latter will give $2a^2\dot{x}x - 4\dot{x}x^3 - 2\dot{z}z = 0$, or $\frac{a^2\dot{x}x - 2\dot{x}x^3}{z} = \dot{z}$, for the Relation of the Celerities $\dot{x}$ and $\dot{z}$. Now $\dot{z}$ being expunged, it will be $2\dot{y}y + \frac{-a^2\dot{x}x + 2\dot{x}x^3}{z} = 0$, and then reſtoring $x\sqrt{aa - xx}$ for z, we ſhall have $2\dot{y}y + \frac{-a^2\dot{x} + 2\dot{x}x^2}{\sqrt{aa - xx}} = 0$, for the Relation between $\dot{x}$ and $\dot{y}$, as was required.

7. Ex. 4. If $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay + xx} = 0$, expreſſes the Relation that is between x and y: I make $\frac{by^3}{a+y} = z$, and $xx\sqrt{ay + xx} = v$, from whence I ſhall have the three Equations $x^3 - ay^2 + z - v = 0$, $az + yz - by^3 = 0$, and $ax^4y + x^6 - vv = 0$. The firſt gives $3\dot{x}x^2 - 2a\dot{y}y + \dot{z} - \dot{v} = 0$; the ſecond gives $a\dot{z} + \dot{z}y + \dot{y}z - 3b\dot{y}y^2 = 0$, and the third gives $4a\dot{x}x^3y + 6\dot{x}x^5 + a\dot{y}x^4 - 2\dot{v}v = 0$, for the Relations of the Velocities $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$. But

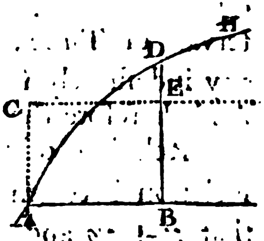
the Values of $\dot{z}$ and $\dot{v}$, found by the ſecond and third Equations, (that is, $\frac{3b\dot{y}y^2-\dot{y}z}{a+y}$ for $\dot{z}$, and $\frac{4a\dot{x}x^3y+6\dot{x}x^5+a\dot{y}x^4}{2v}$ for $\dot{v}$) I ſubſtitute in the firſt Equation, and there ariſes $3\dot{x}x^2-2a\dot{y}y+\frac{3b\dot{y}y^2-\dot{y}z}{a+y}-\frac{4a\dot{x}x^3y-6\dot{x}x^5-a\dot{y}x^4}{2v}=0$. Then inſtead of z and v reſtoring their Values $\frac{by^3}{a+y}$ and $xx\sqrt{ay+xx}$, there will ariſe the Equation ſought $3\dot{x}x^2-2a\dot{y}y+\frac{3ab\dot{y}y^2+2b\dot{y}y^3}{aa+2ay+yy}-\frac{4a\dot{x}xy-6\dot{x}x^3-a\dot{y}xx}{2\sqrt{ay+xx}}=0$, by which the Relation of the Velocities $\dot{x}$ and $\dot{y}$ will be expreſs'd.

8. After what manner the Operation is to be perform'd in other Caſes, I believe is manifeſt from hence; as when in the Equation propos'd there are found ſurd Denominators, Cubick Radicals, Radicals within Radicals, as $\sqrt{ax+\sqrt{aa-xx}}$, or any other complicate Terms of the like kind.

9. Furthermore, altho' in the Equation propoſed there ſhould be Quantities involved, which cannot be determined or expreſs'd by any Geometrical Method, ſuch as Curvilinear Areas or the Lengths of Curve-lines; yet the Relations of their Fluxions may be found, as will appear from the following Example.

Preparation for EXAMPLE 5.

10. Suppoſe BD to be an Ordinate at right Angles to AB, and that ADH be any Curve, which is defined by the Relation between AB and BD exhibited by an Equation. Let AB be called x, and the Area of the Curve ADB, apply'd to Unity, be call'd z. Then erect the Perpendicular AC equal to Unity, and thro' C draw CE parallel to AB, and meeting BD in E. Then conceiving theſe two Superficies ADB and ACEB to be generated by the Motion of the right Line BED; it is manifeſt, that their Fluxions, (that is, the Fluxions of the Quantities $1\times z$, and $1\times x$, or of the Quantities z and x,) are to each other as the generating Lines BD and BE. Therefore $\dot{z}:\dot{x}::BD:BE$ or 1, and therefore $\dot{z}=\dot{x}\times BD$.



11. And hence it is, that z may be involved in any Equation, expreſſing the Relation between x and any other flowing Quantity y; and yet the Relation of the Fluxions $\dot{x}$ and $\dot{y}$ may however be diſcover'd.

12.

12. Ex. 5. As if the Equation $zz + axz - y^4 = 0$ were propos'd to expreſs the Relation between x and y, as alſo $\sqrt{ax - xx} = BD$, for determining a Curve, which therefore will be a Circle. The Equation $zz + axz - y^4 = 0$, as before, will give $2\dot{z}z + a\dot{z}x + a\dot{x}z - 4\dot{y}y^3 = 0$, for the Relation of the Celerities $\dot{x}$, $\dot{y}$, and $\dot{z}$. And therefore ſince it is $\dot{z} = \dot{x} \times BD$ or $= \dot{x}\sqrt{ax - xx}$, ſubſtitute this Value inſtead of it, and there will ariſe the Equation $2\dot{x}z + a\dot{x}x\sqrt{ax - xx} + a\dot{x}z - 4\dot{y}y^3 = 0$, which determines the Relation of the Celerities $\dot{x}$ and $\dot{y}$.

DEMONSTRATION *of the Solution.*

13. The Moments of flowing Quantities, (that is, their indefinitely ſmall Parts, by ~~the acceſſion~~ of which, in indefinitely ſmall portions of Time, they are continually increaſed,) are as the Velocities of their Flowing or Increaſing.

14. Wherefore if the Moment of any one, as x, be repreſented by the Product of its Celerity $\dot{x}$ into an indefinitely ſmall Quantity o (that is, by $\dot{x}o$,) the Moments of the others v, y, z, will be repreſented by $\dot{v}o$, $\dot{y}o$, $\dot{z}o$; becauſe $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$, are to each other as $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$.

15. Now ſince the Moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little acceſſions of the flowing Quantities x and y, by which thoſe Quantities are increaſed through the ſeveral indefinitely little intervals of Time; it follows, that thoſe Quantities x and y, after any indefinitely ſmall interval of Time, become $x + \dot{x}o$ and $y + \dot{y}o$. And therefore the Equation, which at all times indifferently expreſſes the Relation of the flowing Quantities, will as well expreſs the Relation between $x + \dot{x}o$ and $y + \dot{y}o$, as between x and y: So that $x + \dot{x}o$ and $y + \dot{y}o$ may be ſubſtituted in the ſame Equation for thoſe Quantities, inſtead of x and y.

16. Therefore let any Equation $x^3 - ax^2 + axy - y^3 = 0$ be given, and ſubſtitute $x + \dot{x}o$ for x, and $y + \dot{y}o$ for y, and there will ariſe

$$\left.\begin{array}{llll} x^3 & + 3\dot{x}ox^2 & + 3\dot{x}^2oox & + \dot{x}^3o^3 \\ - ax^2 & - 2a\dot{x}ox & - a\dot{x}^2oo & \\ + axy & + a\dot{x}oy & + a\dot{y}ox & + a\dot{x}\dot{y}oo \\ - y^3 & - 3\dot{y}oy^2 & - 3\dot{y}^2ooy & - \dot{y}^3o^3 \end{array}\right\} = 0.$$

17.

17. Now by Suppoſition $x^3 - ax^2 + axy - y^3 = 0$, which therefore being expunged, and the remaining Terms being divided by *o*, there will remain $3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3oo - 2a\dot{x}x - a\dot{x}^2o + a\dot{x}y + a\dot{y}x + a\dot{x}\dot{y}o - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3oo = 0$. But whereas *o* is ſuppoſed to be infinitely little, that it may repreſent the Moments of Quantities; the Terms that are multiply'd by it will be nothing in reſpect of the reſt. Therefore I reject them, and there remains $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$, as above in Examp. 1.

18. Here we may obſerve, that the Terms that are not multiply'd by *o* will always vaniſh, as alſo thoſe Terms that are multiply'd by *o* of more than one Dimenſion. And that the reſt of the Terms being divided by *o*, will always acquire the form that they ought to have by the foregoing Rule: Which was the thing to be proved.

19. And this being now ſhewn, the other things included in the Rule will eaſily follow. As that in the propos'd Equation ſeveral flowing Quantities may be involved; and that the Terms may be multiply'd, not only by the Number of the Dimenſions of the flowing Quantities, but alſo by any other Arithmetical Progreſſions; ſo that in the Operation there may be the ſame difference of the Terms according to any of the flowing Quantities, and the Progreſſion be diſpos'd according to the ſame order of the Dimenſions of each of them. And theſe things being allow'd, what is taught beſides in Examp. 3, 4, and 5, will be plain enough of itſelf.

PROB. II.

An Equation being propoſed, including the Fluxions of Quantities, to find the Relations of thoſe Quantities to one another.

A PARTICULAR SOLUTION.

1. As this Problem is the Converſe of the foregoing, it muſt be ſolved by proceeding in a contrary manner. That is, the Terms multiply'd by $\dot{x}$ being diſpoſed according to the Dimenſions of x; they muſt be divided by $\frac{\dot{x}}{x}$, and then by the number of their Dimenſions, or perhaps by ſome other Arithmetical Progreſſion. Then the ſame work muſt be repeated with the Terms multiply'd by $\dot{v}$, $\dot{y}$,

or $\dot{z}$, and the Sum reſulting muſt be made equal to nothing, rejecting the Terms that are redundant.

2. EXAMPLE. Let the Equation propoſed be $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$. The Operation will be after this manner:

Divide	$3\dot{x}x^2$	$-2a\dot{x}x$	$+a\dot{x}y$	Divide	$-3\dot{y}y^2$	*	$+a\dot{y}x$
by $\frac{\dot{x}}{x}$. Quot.	$3x^3$	$-2ax^2$	$+ayx$	by $\frac{\dot{y}}{y}$. Quot.	$-3y^3$	*	$+axy$
Divide by	3 .	2 .	1.	Divide by	3 .	2 .	1.
Quote	x^3	$-ax^2$	$+ayx$	Quote	$-y^3$	*	$+axy$

Therefore the Sum $x^3 - ax^2 + axy - y^3 = 0$, will be the required Relation of the Quantities x and y. Where it is to be obſerved, that tho' the Term axy occurs twice, yet I do not put it twice in the Sum $x^3 - ax^2 + axy - y^3 = 0$, but I reject the redundant Term. And ſo whenever any Term recurs twice, (or oftener when there are ſeveral flowing Quantities concern'd,) it muſt be wrote only once in the Sum of the Terms.

3. There are other Circumſtances to be obſerved, which I ſhall leave to the Sagacity of the Artiſt; for it would be needleſs to dwell too long upon this matter, becauſe the Problem cannot always be ſolved by this Artifice. I ſhall add however, that after the Relation of the Fluents is obtain'd by this Method, if we can return, by Prob. 1. to the propoſed Equation involving the Fluxions, then the work is right, otherwiſe not. Thus in the Example propoſed, after I have found the Equation $x^3 - ax^2 + axy - y^3 = 0$, if from thence I ſeek the Relation of the Fluxions $\dot{x}$ and $\dot{y}$ by the firſt Problem, I ſhall arrive at the propoſed Equation $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$. Whence it is plain, that the Equation $x^3 - ax^2 + axy - y^3 = 0$ is rightly found. But if the Equation $\dot{x}x - \dot{x}y + a\dot{y} = 0$ were propoſed, by the preſcribed Method I ſhould obtain this $\frac{1}{2}x^2 - xy + ay = 0$, for the Relation between x and y; which Concluſion would be erroneous: Since by Prob. 1. the Equation $\dot{x}x - \dot{x}y - \dot{y}x + a\dot{y} = 0$ would be produced, which is different from the former Equation.

4. Having therefore premiſed this in a perfunctory manner, I ſhall now undertake the general Solution.

A PREPARATION FOR THE GENERAL SOLUTION.

5. Firſt it muſt be obſerved, that in the propoſed Equation the Symbols of the Fluxions, (ſince they are Quantities of a different kind from the Quantities of which they are the Fluxions,) ought to aſcend in every Term to the ſame number of Dimenſions: And when it happens otherwiſe, another Fluxion of ſome flowing Quantity muſt be underſtood to be Unity, by which the lower Terms are ſo often to be multiply'd, till the Symbols of the Fluxions ariſe to the ſame number of Dimenſions in all the Terms. As if the Equation $\dot{x} + \dot{x}\dot{y}x - a\dot{x}\dot{x} = 0$ were propoſed, the Fluxion $\dot{z}$ of ſome third flowing Quantity z muſt be underſtood to be Unity, by which the firſt Term $\dot{x}$ muſt be multiply'd once, and the laſt $a\dot{x}\dot{x}$ twice, that the Fluxions in them may aſcend to as many Dimenſions as in the ſecond Term $\dot{x}\dot{y}x$: As if the propoſed Equation had been derived from this $\dot{x}\dot{z} + \dot{x}\dot{y}x - a\dot{z}\dot{z}\dot{x} = 0$, by putting $\dot{z} = 1$. And thus in the Equation $\dot{y}\dot{x} = \dot{y}\dot{y}$, you ought to imagine $\dot{x}$ to be Unity, by which the Term $\dot{y}\dot{y}$ is multiply'd.

6. Now Equations, in which there are only two flowing Quantities, which every where ariſe to the ſame number of Dimenſions, may always be reduced to ſuch a form, as that on one ſide may be had the Ratio of the Fluxions, (as $\frac{\dot{y}}{\dot{x}}$, or $\frac{\dot{x}}{\dot{y}}$, or $\frac{\dot{z}}{\dot{x}}$, &c.) and on the other ſide the Value of that Ratio, expreſs'd by ſimple Algebraic Terms; as you may ſee here, $\frac{\dot{y}}{\dot{x}} = 2 + 2x - y$. And when the foregoing particular Solution will not take place, it is required that you ſhould bring the Equations to this form.

7. Wherefore when in the Value of that Ratio any Term is denominated by a Compound quantity, or is Radical, or if that Ratio be the Root of an affected Equation; the Reduction muſt be perform'd either by Diviſion, or by Extraction of Roots, or by the Reſolution of an affected Equation, as has been before ſhewn.

8. As if the Equation $\dot{y}a - \dot{y}x - \dot{x}a + \dot{x}x - \dot{x}y = 0$ were propoſed; firſt by Reduction this becomes $\frac{\dot{y}}{\dot{x}} = 1 + \frac{y}{a-x}$, or $\frac{\dot{x}}{\dot{y}} = \frac{a-x}{a-x+y}$. And in the firſt Caſe, if I reduce the Term $\frac{y}{a-x}$, denominated by the compound Quantity $a-x$, to an infinite Series of

ſimple Terms $\frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$ &c. by dividing the Numerator y by the Denominator $a - x$, I ſhall have $\frac{\dot{y}}{\dot{x}} = 1 + \frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$ &c. by the help of which the Relation between x and y is to be determined.

9. So the Equation $\dot{y}\dot{y} = \dot{x}\dot{y} + \dot{x}\dot{x}xx$ being given, or $\frac{\dot{y}\dot{y}}{\dot{x}\dot{x}} = \frac{\dot{y}}{\dot{x}} + xx$, and by a farther Reduction $\frac{\dot{y}}{\dot{x}} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + xx}$: I extract the ſquare Root out of the Terms $\frac{1}{4} + xx$, and obtain the infinite Series $\frac{1}{2} + x^2 - x^4 + 2x^6 - 5x^8 + 14x^{10}$, &c. which if I ſubſtitute for $\sqrt{\frac{1}{4} + xx}$, I ſhall have $\frac{\dot{y}}{\dot{x}} = 1 + x^2 - x^4 + 2x^6 - 5x^8$, &c. or $\frac{\dot{y}}{\dot{x}} = -x^2 + x^4 - 2x^6 + 5x^8$, &c. according as $\sqrt{\frac{1}{4} + xx}$ is either added to $\frac{1}{2}$, or ſubtracted from it.

10. And thus if the Equation $\dot{y}^3 + ax\dot{x}^2\dot{y} + a^2\dot{x}^2\dot{y} - x^3\dot{x}^3 - 2\dot{x}^3a^3 = 0$ were propoſed, or $\frac{\dot{y}^3}{\dot{x}^3} + ax\frac{\dot{y}}{\dot{x}} + a^2\frac{\dot{y}}{\dot{x}} - x^3 - 2a^3 = 0$, I extract the Root of the affected Cubick Equation, and there ariſes $\frac{\dot{y}}{\dot{x}} = a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ &c. as may be ſeen before.

11. But here it may be obſerved, that I look upon thoſe Terms only as compounded, which are compounded in reſpect of flowing Quantities. For I eſteem thoſe as ſimple Quantities which are compounded only in reſpect of given Quantities. For they may be reduced to ſimple Quantities by ſuppoſing them equal to other given Quantities. Thus I conſider the Quantities $\frac{ax+bx}{c}$, $\frac{x}{a+b}$, $\frac{bcc}{ax+bx}$, $\frac{b^4}{ax^2+bx^2}$, $\sqrt{ax+bx}$, &c. as ſimple Quantities, becauſe they may may all be reduced to the ſimple Quantities $\frac{ex}{c}$, $\frac{x}{e}$, $\frac{bc^2}{ex}$, $\frac{b^4}{ex^2}$, $\sqrt{ex}$ (or $e^{\frac{1}{2}}x^{\frac{1}{2}}$) &c. by ſuppoſing $a + b = e$.

12. Moreover, that the flowing Quantities may the more eaſily be diſtinguiſh'd from one another, the Fluxion that is put in the Numerator of the Ratio, or the Antecedent of the Ratio, may not improperly be call'd the *Relate Quantity*, and the other in the Denominator, to which it is compared, the *Correlate*: Alſo the

flowing Quantities may be diſtinguiſh'd by the ſame Names reſpectively. And for the better underſtanding of what follows, you may conceive, that the Correlate Quantity is Time, or rather any other Quantity that flows equably, by which Time is expounded and meaſured. And that the other, or the Relate Quantity, is Space, which the moving Thing, or Point, any how accelerated or retarded, deſcribes in that Time. And that it is the Intention of the Problem, that from the Velocity of the Motion, being given at every Inſtant of Time, the Space deſcribed in the whole Time may be determined.

13. But in reſpect of this Problem Equations may be diſtinguiſh'd into three Orders.

14. Firſt: In which two Fluxions of Quantities, and only one of their flowing Quantities are involved.

15. Second: In which the two flowing Quantities are involved, together with their Fluxions.

16. Third: In which the Fluxions of more than two Quantities are involved.

17. With theſe Premiſes I ſhall attempt the Solution of the Problem, according to theſe three Caſes.

SOLUTION OF CASE I.

18. Suppoſe the flowing Quantity, which alone is contain'd in the Equation, to be the Correlate, and the Equation being accordingly diſpos'd, (that is, by making on one ſide to be only the Ratio of the Fluxion of the other to the Fluxion of this, and on the other ſide to be the Value of this Ratio in ſimple Terms,) multiply the Value of the Ratio of the Fluxions by the Correlate Quantity, then divide each of its Terms by the number of Dimenſions with which that Quantity is there affected, and what ariſes will be equivalent to the other flowing Quantity.

19. So propoſing the Equation $\dot{y}\dot{y} = \dot{x}\dot{y} + \dot{x}\dot{x}xx$; I ſuppoſe x to be the Correlate Quantity, and the Equation being accordingly reduced, we ſhall have $\frac{\dot{y}}{\dot{x}} = 1 + x^2 - x^4 + 2x^6$, &c. Now I multiply the Value of $\frac{\dot{y}}{\dot{x}}$ into x, and there ariſes $x + x^3 - x^5 + 2x^7$, &c. which Terms I divide ſeverally by their number of Dimenſions, and the Reſult $x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7$, &c. I put $= y$. And by

this Equation will be defined the Relation between x and y, as was required.

20. Let the Equation be $\frac{\dot{y}}{\dot{x}} = a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^3}{512a^2}$ &c. there will ariſe $y = ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2}$ &c. for determining the Relation between x and y.

21. And thus the Equation $\frac{\dot{y}}{\dot{x}} = \frac{1}{x^3} - \frac{1}{x^2} + \frac{a}{x^{\frac{1}{2}}} - x^{\frac{1}{2}} + x^{\frac{3}{2}}$, gives $y = -\frac{1}{2x^2} + \frac{1}{x} + 2ax^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}$. For multiply the Value of $\frac{\dot{y}}{\dot{x}}$ into x, and it becomes $\frac{1}{xx} - \frac{1}{x} + ax^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}}$, or $x^{-2} - x^{-1} + ax^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}}$, which Terms being divided by the number of Dimenſions, the Value of y will ariſe as before.

22. After the ſame manner the Equation $\frac{\dot{x}}{\dot{y}} = \frac{2b^2c}{\sqrt{ay^3}} + \frac{3y^2}{a+b} + \sqrt{by+cy}$, gives $x = -\frac{4b^2c}{\sqrt{ay}} + \frac{y^3}{a+b} + \frac{2}{3}\sqrt{by^3+cy^3}$. For the Value of $\frac{\dot{x}}{\dot{y}}$ being multiply'd by y, there ariſes $\frac{2b^2c}{\sqrt{ay}} + \frac{3y^3}{a+b} + \sqrt{by^3+cy^3}$ or $2b^2ca^{-\frac{1}{2}}y^{-\frac{1}{2}} + \frac{3}{a+b}y^3 + \sqrt{b+c} \times y^{\frac{3}{2}}$. And thence the Value of x reſults, by dividing by the number of the Dimenſions of each Term.

23. And ſo $\frac{\dot{y}}{\dot{z}} = z^{\frac{2}{3}}$, gives $y = \frac{3}{5}z^{\frac{5}{3}}$. And $\frac{\dot{y}}{\dot{x}} = \frac{ab}{cx^{\frac{1}{3}}}$, gives $y = \frac{3abx^{\frac{2}{3}}}{2c}$. But the Equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$, gives $y = \frac{a}{0}$. For $\frac{a}{x}$ multiply'd into x makes a, which being divided by the number of Dimenſions, which is 0, there ariſes $\frac{a}{0}$, an infinite Quantity for the Value of y.

24. Wherefore, whenever a like Term ſhall occur in the Value of $\frac{\dot{y}}{\dot{x}}$, whoſe Denominator involves the Correlate Quantity of one Dimenſion only; inſtead of the Correlate Quantity, ſubſtitute the Sum or the Difference between the ſame and ſome other given Quantity to be aſſumed at pleaſure. For there will be the ſame Relation of Flowing, of the Fluents in the Equation ſo produced, as of the Equation at firſt propoſed; and the infinite Relate Quantity

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tity by this means will be diminiſh'd by an infinite part of itſelf, and will become finite, but yet conſiſting of Terms infinite in number.

25. Therefore the Equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$ being propoſed, if for x I write $b+x$, aſſuming the Quantity b at pleaſure, there will ariſe $\frac{\dot{y}}{\dot{x}} = \frac{a}{b+x}$; and by Diviſion $\frac{\dot{y}}{\dot{x}} = \frac{a}{b} - \frac{ax}{b^2} + \frac{ax^2}{b^3} - \frac{ax^3}{b^4}$ &c. And now the Rule aforegoing will give $y = \frac{ax}{b} - \frac{ax^2}{2b^2} + \frac{ax^3}{3b^3} - \frac{ax^4}{4b^4}$ &c. for the Relation between x and y.

26. So if you have the Equation $\frac{\dot{y}}{\dot{x}} = \frac{2}{x} + 3 - xx$; becauſe of the Term $\frac{2}{x}$, if you write $1+x$ for x, there will ariſe $\frac{\dot{y}}{\dot{x}} = \frac{2}{1+x} + 2 - 2x - xx$. Then reducing the Term $\frac{2}{1+x}$ into an infinite Series $+2 - 2x + 2x^2 - 2x^3 + 2x^4$, &c. you will have $\frac{\dot{y}}{\dot{x}} = 4 - 4x + x^2 - 2x^3 + 2x^4$, &c. And then according to the Rule $y = 4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{2}{5}x^5$, &c. for the Relation of x and y.

27. And thus if the Equation $\frac{\dot{y}}{\dot{x}} = x^{-\frac{1}{2}} + x^{-1} - x^{\frac{1}{2}}$ were propoſed; becauſe I here obſerve the Term x^{-1} (or $\frac{1}{x}$) to be found, I tranſmute x, by ſubſtituting $1-x$ for it, and there ariſes $\frac{\dot{y}}{\dot{x}} = \frac{1}{\sqrt{1-x}} + \frac{1}{1-x} - \sqrt{1-x}$. Now the Term $\frac{1}{1-x}$ produces $1 + x + x^2 + x^3$, &c. and the Term $\sqrt{1-x}$ is equivalent to $1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$, and therefore $\frac{1}{\sqrt{1-x}}$ or $\frac{1}{1-\frac{1}{2}x-\frac{1}{8}x^2,\ \&c.}$ is the ſame as $1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{8}x^3$, &c. So that when theſe Values are ſubſtituted, I ſhall have $\frac{\dot{y}}{\dot{x}} = 1 + 2x + \frac{3}{2}x^2 + \frac{17}{16}x^3$, &c. And then by the Rule $y = x + x^2 + \frac{1}{2}x^3 + \frac{17}{64}x^4$, &c. And ſo of others.

28. Alſo in other Caſes the Equation may ſometimes be conveniently reduced, by ſuch a Tranſmutation of the flowing Quantity. As if this Equation were propoſed $\frac{\dot{y}}{\dot{x}} = \frac{c^2x}{c^3 - 3c^2x + 3cx^2 - x^3}$; inſtead

of x I write $c - x$, and then I ſhall have $\frac{\dot{y}}{\dot{x}} = \frac{c^3 - c^2x}{x^3}$ or $\frac{c^3}{x^3} - \frac{c^2}{x^2}$; and then by the Rule $y = -\frac{c^3}{2xx} + \frac{c^2}{x}$. But the uſe of ſuch Tranſmutations will appear more plainly in what follows.

SOLUTION OF CASE II.

29. PREPARATION. And ſo much for Equations that involve only one Fluent. But when each of them are found in the Equation, firſt it muſt be reduced to the Form preſcribed, by making, that on one ſide may be had the Ratio of the Fluxions, equal to an aggregate of ſimple Terms on the other ſide.

30. And beſides, if in the Equations ſo reduced there be any Fractions denominated by the flowing Quantity, they muſt be freed from thoſe Denominators, by the above-mentioned Tranſmutation of the flowing Quantity.

31. So the Equation $\dot{y}ax - \dot{x}xy - aa\dot{x} = 0$ being propoſed, or $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{x}$; becauſe of the Term $\frac{a}{x}$, I aſſume b at pleaſure, and for x I either write $b + x$, or $b - x$, or $x - b$. As if I ſhould write $b + x$, it will become $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{b + x}$. And then the Term $\frac{a}{b + x}$ being converted by Diviſion into an infinite Series, we ſhall have $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{b} - \frac{ax}{b^2} + \frac{ax^2}{b^3} - \frac{ax^3}{b^4}$, &c.

32. And after the ſame manner the Equation $\frac{\dot{y}}{\dot{x}} = 3y - 2x + \frac{x}{y} - \frac{2y}{xx}$ being propoſed; if, by reaſon of the Terms $\frac{x}{y}$ and $\frac{2y}{xx}$, I write $1 - y$ for y, and $1 - x$ for x, there will ariſe $\frac{\dot{y}}{\dot{x}} = 1 - 3y + 2x + \frac{1 - x}{1 - y} + \frac{2y - 2}{1 - 2x + x^2}$. But the Term $\frac{1 - x}{1 - y}$ by infinite Diviſion gives $1 - x + y - xy + y^2 - xy^2 + y^3 - xy^3$, &c. and the Term $\frac{2y - 2}{1 - 2x + xx}$ by a like Diviſion gives $2y - 2 + 4xy - 4x + 6x^2y - 6x^2 + 8x^3y - 8x^3 + 10x^4y - 10x^4$, &c. Therefore $\frac{\dot{y}}{\dot{x}} = -3x + 3xy + y^2 - xy^2 + y^3 - xy^3$, &c. $+ 6x^2y - 6x^2 + 8x^3y - 8x^3 + 10x^4y - 10x^4$, &c.

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33. RULE. The Equation being thus prepared, when need requires, diſpoſe the Terms according to the Dimenſions of the flowing Quantities, by ſetting down firſt thoſe that are not affected by the Relate Quantity, then thoſe that are affected by its leaſt Dimenſion, and ſo on. In like manner alſo diſpoſe the Terms in each of theſe Claſſes according to the Dimenſions of the other Correlate Quantity, and thoſe in the firſt Claſs, (or ſuch as are not affected by the Relate Quantity,) write in a collateral order, proceeding towards the right hand, and the reſt in a deſcending Series in the left-hand Column, as the following Diagrams indicate. The work being thus prepared, multiply the firſt or the loweſt of the Terms in the firſt Claſs by the Correlate Quantity, and divide by the number of Dimenſions, and put this in the Quote for the initial Term of the Value of the Relate Quantity. Then ſubſtitute this into the Terms of the Equation that are diſpoſed in the left-hand Column, inſtead of the Relate Quantity, and from the next loweſt Terms you will obtain the ſecond Term of the Quote, after the ſame manner as you obtain'd the firſt. And by repeating the Operation you may continue the Quote as far as you pleaſe. But this will appear plainer by an Example or two.

34. EXAMP. I. Let the Equation $\frac{\dot{y}}{x} = 1 - 3x + y + x^2 + xy$ be propoſed, whoſe Terms $1 - 3x + x^2$, which are not affected by the Relate Quantity y, you ſee diſpos'd collaterally in the uppermoſt Row, and the reſt y and xy in the left-hand Column. And firſt I multiply the initial Term 1 into the Correlate Quantity x, and it makes x, which being divided by the number of Dimenſions 1, I place it in the Quote under-written. Then ſubſtituting this Term inſtead of y in the marginal Terms $+y$ and $+xy$, I have $+x$ and $+xx$, which I write over againſt them to the right hand. Then from the reſt I take the loweſt Terms $-3x$ and $+x$, whoſe aggregate $-2x$ multiply'd into x becomes $-2xx$, and

	$+1 - 3x + xx$
$+y$	$* + x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5$, &c.
$+xy$	$* \quad * + xx - x^3 + \frac{1}{3}x^4 - \frac{1}{6}x^5 + \frac{1}{30}x^6$, &c
The Sum	$1 - 2x + xx - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{4}{30}x^5$, &c.
$y =$	$x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6$, &c.

being divided by the number of Dimenſions 2, gives $-xx$ for the ſecond Term of the Value of y in the Quote. Then this Term being likewiſe aſſumed to compleat the Value of the Marginals $+y$ and $+xy$, there will ariſe alſo $-xx$ and $-x^3$, to be added to the Terms $+x$ and $+xx$ that were before inſerted. Which being done, I again aſſume the next loweſt Terms $+xx$, $-xx$, and $+xx$, which I collect into one Sum xx, and thence I derive (as before) the third Term $+\frac{1}{3}x^3$, to be put in the Value of y. Again, taking this Term $\frac{1}{3}x^3$ into the Values of the marginal Terms, from the next loweſt $+\frac{1}{3}x^3$ and $-x^3$ added together, I obtain $-\frac{1}{6}x^4$ for the fourth Term of the Value of y. And ſo on *in infinitum.*

35. EXAMP. 2. In like manner if it were required to determine the Relation of x and y in this Equation, $\frac{\dot{y}}{\dot{x}} = 1 + \frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{4}$, &c. which Series is ſuppoſed to proceed *ad infinitum*; I put 1 in the beginning, and the other Terms in the left-hand Column, and then purſue the work according to the following Diagram.

	$+1$						
$+\frac{y}{a}$	$*$	$+\frac{x}{a}$	$+\frac{x^2}{2a^2}$	$+\frac{x^3}{2a^3}$	$+\frac{x^4}{2a^4}$	$+\frac{x^5}{2a^5}$,	&c.
$+\frac{xy}{a^2}$	$*$	$*$	$\frac{x^2}{a^2}$	$+\frac{x^3}{2a^3}$	$+\frac{x^4}{2a^4}$	$+\frac{x^5}{2a^5}$,	&c.
$+\frac{x^2y}{a^3}$	$*$	$*$	$*$	$+\frac{x^3}{a^3}$	$+\frac{x^4}{2a^4}$	$+\frac{x^5}{2a^5}$,	&c.
$+\frac{x^3y}{a^4}$	$*$	$*$	$*$	$*$	$+\frac{x^4}{a^4}$	$+\frac{x^5}{2a^5}$,	&c.
$+\frac{x^4y}{a^5}$	$*$	$*$	$*$	$*$	$*$	$+\frac{x^5}{a^5}$,	&c.
&c.							
Sum	1	$+\frac{x}{a}$	$+\frac{3x^2}{2a^2}$	$+\frac{2x^3}{a^3}$	$+\frac{5x^4}{2a^4}$	$+\frac{3x^5}{a^5}$,	&c.
$y=$	x	$+\frac{x^2}{2a}$	$+\frac{x^3}{2a^2}$	$+\frac{x^4}{2a^3}$	$+\frac{x^5}{2a^4}$	$+\frac{x^6}{2a^5}$,	&c.

36. As I here propoſed to extract the Value of y as far as ſix Dimenſions of x only; for that reaſon I omit all the Terms in the Operation which I foreſee will contribute nothing to my purpoſe, as is intimated by the Mark, *&c.* which I have ſubjoin'd to the Series that are cut off.

37. **EXAMP. 3.** In like manner if this Equation were propoſed $\frac{\dot{y}}{\dot{x}} = -3x + 3xy + y^2 - xy^2 + y^3 - xy^3 + y^4 - xy^4$, &c. $+ 6x^2y - 6x^2 + 8x^3y - 8x^3 + 10xy^4 - 10x^4$, &c. and it is intended to extract the Value of y as far as ſeven Dimenſions of x. I place the Terms in order, according to the following Diagram, and I work as before, only with this exception, that ſince in the left-hand Column y is not only of one, but alſo of two and three Dimenſions; (or of more than three, if I intended to produce the Value of y beyond the degree of x^7,) I ſubjoin the ſecond and third Powers of the Value of y, ſo far gradually produced, that when they are ſubſtituted by degrees to the right-hand, in the Values of the Marginals to the left, Terms may ariſe of ſo many Dimenſions as I obſerve to be required for the following Operation. And by this Method there ariſes at length $y = -\frac{3}{2}x^2 - 6x^3 - \frac{25}{8}x^4$, &c. which is the

	$-3x$	$-6x^2$	$-8x^3$	$-10x^4$	$-12x^5$	$-14x^6$, &c.
$+3xy$	*	*	$-\frac{9}{2}x^3$	$-6x^4$	$-\frac{75}{8}x^5$	$-\frac{273}{20}x^6$, &c.
$+6x^2y$	*	*	*	$-9x^4$	$-12x^5$	$-\frac{75}{4}x^6$, &c.
$+8x^3y$	*	*	*	*	$-12x^5$	$-16x^6$, &c.
$+10x^4y$	*	*	*	*	*	$-15x^6$, &c.
&c.						
$+y^2$	*	*	*	$+\frac{9}{4}x^4$	$+6x^5$	$+\frac{107}{8}x^6$, &c.
$-xy^2$	*	*	*	*	$-\frac{9}{4}x^5$	$-6x^6$, &c.
&c.						
$+y^3$	*	*	*	*	*	$-\frac{27}{8}x^6$, &c.
Sum	$-3x$	$-6x^2$	$-\frac{25}{2}x^3$	$-\frac{91}{4}x^4$	$-\frac{333}{8}x^5$	$-\frac{367}{5}x^6$, &c.
$y =$	$-\frac{3}{2}x^2$	$-2x^3$	$-\frac{25}{8}x^4$	$-\frac{91}{20}x^5$	$-\frac{111}{16}x^6$	$-\frac{367}{35}x^7$, &c.
$y^2 =$	$+\frac{9}{4}x^4$	$+6x^5$	$+\frac{107}{8}x^6$, &c.			
$y^3 =$	$-\frac{27}{8}x^6$, &c.					

Equation required. But whereas this Value is negative, it appears that one of the Quantities x or y decreaſes, while the other increaſes. And the ſame thing is alſo to be concluded, when one of the Fluxions is affirmative, and the other negative.

38. EXAMP. 4. You may proceed in like manner to reſolve the Equation, when the Relate Quantity is affected with fractional Dimenſions. As if it were propoſed to extract the Value of x from this Equation, $\frac{\dot{x}}{\dot{y}} = \frac{1}{2}y - 4y^2 + 2yx^{\frac{1}{2}} - \frac{4}{5}x^2 + 7y^{\frac{5}{2}} + 2y^3$, in

	$+\frac{1}{2}y \quad * \quad -4y^2 + 7y^{\frac{5}{2}} + 2y^3$
$2yx^{\frac{1}{2}}$ $-\frac{4}{5}x^2$	$* \quad * \quad +y^2 \quad * \quad -2y^3 + 4y^{\frac{7}{2}} - 2y^4$, &c. $* \quad * \quad * \quad * \quad * \quad * \quad -\frac{1}{20}y^4$, &c.
Sum	$+\frac{1}{2}y \quad * \quad -3y^2 + 7y^{\frac{5}{2}} \quad * \quad +4y^{\frac{7}{2}} - \frac{41}{20}y^4$, &c.
$x =$ $x^{\frac{1}{2}} =$ $x^2 =$	$+\frac{1}{4}y^2 - y^3 + 2y^{\frac{7}{2}} \quad * \quad +\frac{8}{9}y^{\frac{9}{2}} - \frac{41}{100}y^5$, &c. $+\frac{1}{2}y - y^2 + 2y^{\frac{5}{2}} - y^3$, &c. $\frac{1}{16}y^4$, &c.

which x in the Term $2yx^{\frac{1}{2}}$ (or $2y\sqrt{x}$) is affected with the Fractional Dimenſion $\frac{1}{2}$. From the Value of x I derive by degrees the Value of $x^{\frac{1}{2}}$, (that is, by extracting its ſquare-Root,) as may be obſerved in the lower part of this Diagram; that it may be inſerted and transfer'd gradually into the Value of the marginal Term $2yx^{\frac{1}{2}}$. And ſo at laſt I ſhall have the Equation $x = \frac{1}{4}y^2 - y^3 + 2y^{\frac{7}{2}} + \frac{8}{9}y^{\frac{9}{2}} - \frac{41}{100}y^5$, &c. by which x is expreſs'd indefinitely in reſpect of y. And thus you may operate in any other caſe whatſoever.

39. I ſaid before, that theſe Solutions may be perform'd by an infinite variety of ways. This may be done if you aſſume at pleaſure not only the initial quantity of the upper Series, but any other given quantity for the firſt Term of the Quote, and then you may proceed as before. Thus in the firſt of the preceding Examples, if you aſſume 1 for the firſt Term of the Value of y, and ſubſtitute it for y in the marginal Terms $+y$ and $+xy$, and purſue the reſt of the Operation as before, (of which I have here given a

	$+1-3x+xx$
$+y$	$+1+2x \quad * \quad +x^3+\frac{1}{4}x^4$, &c.
$+xy$	$* \; +x \; +2x^2 \quad * \quad +x^4$, &c.
Sum	$+2 \quad * \quad +3x^2+x^3+\frac{5}{4}x^4$, &c.
$y=$	$1+2x \quad * \quad +x^3+\frac{1}{4}x^4+\frac{1}{4}x^5$, &c.

Specimen,) another Value of y will ariſe, $1+2x+x^3+\frac{1}{4}x^4$, &c. And thus another and another Value may be produced, by aſſuming 2, or 3, or any other number for its firſt Term. Or if you make uſe of any Symbol, as a, to repreſent the firſt Term indefinitely, by the ſame method of Operation, (which I ſhall here ſet down,) you will find $y=a+x+ax-xx+axx+\frac{1}{3}x^3+\frac{2}{3}ax^3$, &c. which being found, for a you may ſubſtitute 1, 2, 0, $\frac{1}{2}$, or any other Number, and thereby obtain the Relation between x and y an infinite variety of ways.

	$+1-3x+xx$
$+y$	$+a+x \quad -xx \quad +\frac{1}{3}x^3$, &c. $+ax \quad +ax^2 \quad +\frac{2}{3}ax^3$, &c.
$+xy$	$* \; +ax \quad +x^2 \quad -x^3$, &c. $+ax^2 \quad +ax^3$, &c.
Sum	$+1-2x \quad +x^2 \quad -\frac{2}{3}x^3$, &c. $+a+2ax+2ax^2+\frac{5}{3}ax^3$, &c.
$y=$	$a+x \quad -x^2 \quad +\frac{1}{3}x^3 \quad -\frac{1}{6}x^4$, &c. $+ax \quad +ax^2 \quad +\frac{2}{3}ax^3 \quad +\frac{5}{12}ax^4$, &c.

40. And it is to be obſerved, that when the Quantity to be extracted is affected with a Fractional Dimenſion, (as you ſee in the fourth of the preceding Examples,) then it is convenient to take Unity, or ſome other proper Number, for its firſt Term. And indeed this is neceſſary, when to obtain the Value of that fractional Dimenſion, the Root cannot otherwiſe be extracted, becauſe of the negative Sign; as alſo when there are no Terms to be diſpoſed in the firſt or capital Claſs, from which that initial Term may be deduced.

41. And thus at laſt I have compleated this moſt troubleſome and of all others moſt difficult Problem, when only two flowing Quantities, together with their Fluxions, are comprehended in an Equation. But beſides this general Method, in which I have taken in all the Difficulties, there are others which are generally ſhorter, by which the Work may often be eaſed; to give ſome Specimens of which, *ex abundanti*, perhaps will not be diſagreeable to the Reader.

42. I. If it happen that the Quantity to be reſolved has in ſome places negative Dimenſions, it is not of abſolute neceſſity that therefore the Equation ſhould be reduced to another form. For thus the Equation $\dot{y} = \frac{1}{y} - xx$ being propoſed, where y is of one negative Dimenſion, I might indeed reduce it to another Form, as by writing $1 + y$ for y; but the Reſolution will be more expedite as you have it in the following Diagram.

	$* \quad * - xx$
$\frac{1}{y}$	$1 - x + \frac{3}{2}xx$, &c.
Sum	$1 - x + \frac{1}{2}xx$
$y =$	$1 + x - \frac{1}{2}xx + \frac{1}{6}x^3$, &c.
$\frac{1}{y} =$	$1 - x + \frac{3}{2}xx$, &c.

43. Here aſſuming 1 for the initial Term of the Value of y, I extract the reſt of the Terms as before, and in the mean time I deduce from thence, by degrees, the Value of $\frac{1}{y}$ by Diviſion, and inſert it in the Value of the marginal Term.

44. II. Neither is it neceſſary that the Dimenſions of the other flowing Quantity ſhould be always affirmative. For from the Equation $\dot{y} = 3 + 2y - \frac{yy}{x}$, without the preſcribed Reduction of the Term $\frac{yy}{x}$, there will ariſe $y = 3x - \frac{3}{2}xx + 2x^3$, &c.

45. And from the Equation $\dot{y} = -y + \frac{1}{x} - \frac{1}{xx}$, the Value of y will be found $y = \frac{1}{x}$, if the Operation be perform'd after the Manner of the following Specimen.

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	$-\frac{1}{xx}+\frac{1}{x}$
$-y$	$* \quad -\frac{1}{x}$
Sum	$-\frac{1}{xx} \quad 0$
$y=$	$\frac{1}{x}$

46. Here we may obſerve by the way, that among the infinite manners by which any Equation may be reſolved, it often happens that there are ſome, that terminate at a finite Value of the Quantity to be extracted, as in the foregoing Example. And theſe are not difficult to find, if ſome Symbol be aſſumed for the firſt Term. For when the Reſolution is perform'd, then ſome proper Value may be given to that Symbol, which may render the whole finite.

47. III. Again, if the Value of y is to be extracted from this Equation $y = \frac{y}{2x} + 1 - 2x + \frac{1}{2}xx$, it may be done conveniently enough, without any Reduction of the Term $\frac{y}{2x}$, by ſuppoſing (after the manner of Analyſts,) that to be given which is required. Thus for the firſt Term of the Value of y I put $2ex$, taking $2e$ for the numeral Coefficient which is yet unknown. And ſubſtituting $2ex$ inſtead of y, in the marginal Term, there ariſes e, which I write on the right-hand; and the Sum $1+e$ will give $x+ex$ for the ſame firſt Term of the Value of y, which I had firſt repreſented by the Term $2ex$. Therefore I make $2ex = x + ex$, and thence I deduce $e = 1$. So that the firſt Term $2ex$ of the Value of y is $2x$. After the ſame manner I make uſe of the fictitious Term $2fx^2$ to repreſent the ſecond Term of the Value of y, and thence at laſt I derive $-\frac{2}{3}$ for the Value of f, and therefore that ſecond Term is $-\frac{4}{3}xx$. And ſo the fictitious Coefficient g in the third Term will give $\frac{1}{10}$, and h in the fourth Term will be 0. Wherefore ſince there are no other Terms remaining, I conclude the work is finiſh'd, and that the Value of y is exactly $2x - \frac{4}{3}x^2 + \frac{1}{5}x^3$. See the Operation in the following Diagram.

	$1 - 2x + \frac{1}{2}xx$
$\frac{y}{2x}$	$e + fx + gxx + hx^3$
Sum.	$+1 - 2x + \frac{1}{2}xx$ $+e + fx + gx^2 + hx^3$
Hypothetically $y =$	$2ex + 2fx^2 + 2gx^3 + 2hx^4$
	‖ ‖ ‖ ‖
Conſequentially $y =$	$+x - x^2 + \frac{1}{6}x^3 + \frac{1}{4}hx^4$ $+ex + \frac{1}{2}fx^2 + \frac{1}{3}gx^3$
Real Value $y =$	$2x - \frac{4}{3}x^2 + \frac{1}{5}x^3$

48. Much after the ſame manner, if it were $\dot{y} = \frac{3y}{4x}$; ſuppoſe $y = ex^s$, where e denotes the unknown Coefficient, and s the number of Dimenſions, which is alſo unknown. And ex^s being ſubſtituted for y, there will ariſe $\dot{y} = \frac{3ex^{s-1}}{4}$, and thence again $y = \frac{3ex^s}{4s}$. Compare theſe two Values of y, and you will find $\frac{3e}{4s} = e$, and therefore $s = \frac{3}{4}$, and e will be indefinite. Therefore aſſuming e at pleaſure, you will have $y = ex^{\frac{3}{4}}$.

49. IV. Sometimes alſo the Operation may be begun from the higheſt Dimenſion of the equable Quantity, and continually proceed to the lower Powers. As if this Equation were given, $\dot{y} = \frac{y}{xx} + \frac{1}{xx} + 3 + 2x - \frac{4}{x}$, and we would begin from the higheſt Term $2x$, by diſpoſing the capital Series in an order contrary to the foregoing; there will ariſe at laſt $y = xx + 4x - \frac{1}{x}$, &c. as may be ſeen in the form of working here ſet down.

	$+2x + 3 - \frac{4}{x} + \frac{1}{xx}$
$+\frac{y}{xx}$	$* + 1 + \frac{4}{x} * - \frac{1}{x^3} + \frac{1}{2x^4}$, &c.
Sum	$+2x + 4 * + \frac{1}{xx} - \frac{1}{x^3} + \frac{1}{2x^4}$, &c.
$y =$	$x^2 + 4x * - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3}$, &c.

50. And here it may be obſerved by the way, that as the Operation proceeded, I might have inſerted any given Quantity between the Terms $4x$ and $-\frac{1}{x}$, for the intermediate Term that is deficient, and ſo the Value of y might have been exhibited an infinite variety of ways.

51. V. If there are beſides any fractional Indices of the Dimenſions of the Relate Quantity, they may be reduced to Integers by ſuppoſing that Quantity, which is affected by its fractional Dimenſion, to be equal to any third Fluent; and then by ſubſtituting that Quantity, as alſo its Fluxion, ariſing from that fictitious Equation, inſtead of the Relate Quantity and its Fluxion.

52. As if the Equation $\dot{y} = 3xy^{\frac{2}{3}} + y$ were propoſed, where the Relate Quantity is affected with the fractional Index $\frac{2}{3}$ of its Dimenſion; a Fluent z being aſſumed at pleaſure, ſuppoſe $y^{\frac{1}{3}} = z$, or $y = z^3$; the Relation of the Fluxions, by Prob. I. will be $\dot{y} = 3\dot{z}z^2$. Therefore ſubſtituting $3\dot{z}z^2$ for $\dot{y}$, as alſo z^3 for y, and z^2 for $y^{\frac{2}{3}}$, there will ariſe $3\dot{z}z^2 = 3xz^2 + z^3$, or $\dot{z} = x + \frac{1}{3}z$, where z performs the office of the Relate Quantity. But after the Value of z is extracted, as $z = \frac{1}{2}x^2 + \frac{x^3}{18} + \frac{x^4}{216} + \frac{x^5}{3240}$, &c. inſtead of z reſtore $y^{\frac{1}{3}}$, and you will have the deſired Relation between x and y; that is, $y^{\frac{1}{3}} = \frac{1}{2}x^2 + \frac{1}{18}x^3 + \frac{1}{216}x^4$, &c. and by Cubing each ſide, $y = \frac{1}{8}x^6 + \frac{1}{24}x^7 + \frac{1}{288}x^8$, &c.

53. In like manner if the Equation $\dot{y} = \surd 4y + \surd xy$ were given, or $\dot{y} = 2y^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}}$; I make $z = y^{\frac{1}{2}}$, or $zz = y$, and thence by Prob. I. $2\dot{z}z = \dot{y}$, and by conſequence $2\dot{z}z = 2z + x^{\frac{1}{2}}z$, or $\dot{z} = 1 + \frac{1}{2}x^{\frac{1}{2}}$. Therefore by the firſt Caſe of this 'tis $z = x + \frac{1}{3}x^{\frac{3}{2}}$, or $y^{\frac{1}{2}} = x + \frac{1}{3}x^{\frac{3}{2}}$, then by ſquaring each ſide, $y = xx + \frac{2}{3}x^{\frac{5}{2}} + \frac{1}{9}x^3$. But if you ſhould deſire to have the Value of y exhibited an infinite number of ways, make $z = c + x + \frac{1}{3}x^{\frac{3}{2}}$, aſſuming any initial Term c, and it will be zz, that is y, $= c^2 + 2cx + \frac{2}{3}cx^{\frac{3}{2}} + x^2 + \frac{2}{3}x^{\frac{5}{2}} + \frac{1}{9}x^3$. But perhaps I may ſeem too minute, in treating of ſuch things as will but ſeldom come into practice.

SOLUTION OF CASE III.

54. The Reſolution of the Problem will ſoon be diſpatch'd, when the Equation involves three or more Fluxions of Quantities. For

between any two of thoſe Quantities any Relation may be aſſumed, when it is not determined by the State of the Queſtion, and the Relation of their Fluxions may be found from thence; ſo that either of them, together with its Fluxion, may be exterminated. For which reaſon if there are found the Fluxions of three Quantities, only one Equation need to be aſſumed, two if there be four, and ſo on; that the Equation propos'd may finally be transform'd into another Equation, in which only two Fluxions may be found. And then this Equation being reſolved as before, the Relations of the other Quantities may be diſcover'd.

55. Let the Equation propoſed be $2\dot{x} - \dot{z} + \dot{y}x = 0$; that I may obtain the Relation of the Quantities x, y, and z, whoſe Fluxions $\dot{x}$, $\dot{y}$, and $\dot{z}$ are contained in the Equation; I form a Relation at pleaſure between any two of them, as x and y, ſuppoſing that $x = y$, or $2y = a + z$, or $x = yy$, &c. But ſuppoſe at preſent $x = yy$, and thence $\dot{x} = 2\dot{y}y$. Therefore writing $2\dot{y}y$ for $\dot{x}$, and yy for x, the Equation propoſed will be transform'd into this: $4\dot{y}y - \dot{z} + \dot{y}y^2 = 0$. And thence the Relation between y and z will ariſe, $2yy + \frac{1}{3}y^3 = z$. In which if x be written for yy, and $x^{\frac{3}{2}}$ for y^3, we ſhall have $2x + \frac{1}{3}x^{\frac{3}{2}} = z$. So that among the infinite ways in which x, y, and z may be related to each other, one of them is here found, which is repreſented by theſe Equations, $x = yy$, $2y^2 + \frac{1}{3}y^3 = z$, and $2x + \frac{1}{3}x^{\frac{3}{2}} = z$.

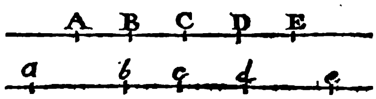
DEMONSTRATION.

56. And thus we have ſolved the Problem, but the Demonſtration is ſtill behind. And in ſo great a variety of matters, that we may not derive it ſynthetically, and with too great perplexity, from its genuine foundations, it may be ſufficient to point it out thus in ſhort, by way of Analyſis. That is, when any Equation is propos'd, after you have finiſh'd the work, you may try whether from the derived Equation you can return back to the Equation propos'd, by Prob. 1. And therefore, the Relation of the Quantities in the derived Equation requires the Relation of the Fluxions in the propoſed Equation, and contrary-wiſe: which was to be ſhewn.

57. So if the Equation propoſed were $\dot{y} = x$, the derived Equation will be $y = \frac{1}{2}x^2$; and on the contrary, by Prob. 1. we have $\dot{y} = \dot{x}x$, that is, $\dot{y} = x$, becauſe $\dot{x}$ is ſuppoſed Unity. And thus

from $\dot{y} = 1 - 3x + y + xx + xy$ is derived $y = x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6$, &c. And thence by Prob. 1. $\dot{y} = 1 - 2x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{2}{15}x^5$, &c. Which two Values of $\dot{y}$ agree with each other, as appears by ſubſtituting $x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5$, &c. inſtead of y in the firſt Value.

58. But in the Reduction of Equations I made uſe of an Operation, of which alſo it will be convenient to give ſome account. And that is, the Tranſmutation of a flowing Quantity by its connexion with a given Quantity. Let AE and *ae* be two Lines indefinitely extended each way, along which two moving Things or Points may paſs from afar, and at the ſame time may reach the places A and *a*, B and *b*, C and *c*, D and *d*, &c. and let B be the Point, by its diſtance from which, the Motion of the moving thing or point in AE is eſtimated; ſo that — BA, BC, BD, BE, ſucceſſively, may be the flowing Quantities, when the moving thing is in the places A, C, D, E. Likewiſe let *b* be a like point in the other Line. Then will — BA and — *ba* be contemporaneous Fluents, as alſo BC and *bc*, BD and *bd*, BE and *be*, &c. Now if inſtead of the points B and *b*, be ſubſtituted A and *c*, to which, as at reſt, the Motions are refer'd; then o and — *ca*, AB and — *cb*, AC and o, AD and *cd*, AE and *ce*, will be contemporaneous flowing Quantities. Therefore the flowing Quantities are changed by the Addition and Subtraction of the given Quantities AB and *ac*; but they are not changed as to the Celerity of their Motions, and the mutual reſpect of their Fluxion. For the contemporaneous parts AB and *ab*, BC and *bc*, CD and *cd*, DE and *de*, are of the ſame length in both caſes. And thus in Equations in which theſe Quantities are repreſented, the contemporaneous parts of Quantities are not therefore changed, notwithſtanding their abſolute magnitude may be increaſed or diminiſhed by ſome given Quantity. Hence the thing propoſed is manifeſt: For the only Scope of this Problem is, to determine the contemporaneous Parts, or the contemporary Differences of the abſolute Quantities *v*, *x*, *y*, or *z*, deſcribed with a given Rate of Flowing. And it is all one of what abſolute magnitude thoſe Quantities are, ſo that their contemporary or correſpondent Differences may agree with the propoſed Relation of the Fluxions.



59. The reaſon of this matter may alſo be thus explain'd Algebraically. Let the Equation $y = xxy$ be propoſed, and ſup-

poſe $x = 1 + z$. Then by Prob. 1. $\dot{x} = \dot{z}$. So that for $\dot{y} = \dot{x}xy$, may be wrote $\dot{y} = \dot{x}y + \dot{x}zy$. Now ſince $\dot{x} = \dot{z}$, it is plain, that though the Quantities x and z be not of the ſame length, yet that they flow alike in reſpect of y, and that they have equal contemporaneous parts. Why therefore may I not repreſent by the ſame Symbols Quantities that agree in their Rate of Flowing; and to determine their contemporaneous Differences, why may not I uſe $\dot{y} = \dot{x}y + \dot{x}xy$ inſtead of $\dot{y} = \dot{x}xy$?

60. Laſtly it appears plainly in what manner the contemporary parts may be found, from an Equation involving flowing Quantities. Thus if $y = \frac{1}{x} + x$ be the Equation, when $x = 2$, then $y = 2\frac{1}{2}$. But when $x = 3$, then $y = 3\frac{1}{3}$. Therefore while x flows from 2 to 3, y will flow from $2\frac{1}{2}$ to $3\frac{1}{3}$. So that the parts deſcribed in this time are $3 - 2 = 1$, and $3\frac{1}{3} - 2\frac{1}{2} = \frac{5}{6}$.

61. This Foundation being thus laid for what follows, I ſhall now proceed to more particular Problems.

PROB. III.

To determine the Maxima *and* Minima *of Quantities.*

1. When a Quantity is the greateſt or the leaſt that it can be, at that moment it neither flows backwards or forwards. For if it flows forwards, or increaſes, that proves it was leſs, and will preſently be greater than it is. And the contrary if it flows backwards, or decreaſes. Wherefore find its Fluxion, by Prob. 1. and ſuppoſe it to be nothing.

2. EXAMP. I. If in the Equation $x^3 - ax^2 + axy - y^3 = 0$ the greateſt Value of x be required; find the Relation of the Fluxions of x and y, and you will have $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$. Then making $\dot{x} = 0$, there will remain $- 3\dot{y}y^2 + a\dot{y}x = 0$, or $3y^2 = ax$. By the help of this you may exterminate either x or y out of the primary Equation, and by the reſulting Equation you may determine the other, and then both of them by $- 3y^2 + ax = 0$.

3. This Operation is the ſame, as if you had multiply'd the Terms of the propoſed Equation by the number of the Dimenſions of the other flowing Quantity y. From whence we may derive the

famous Rule of *Huddenius*, that, in order to obtain the greateſt or leaſt Relate Quantity, the Equation muſt be diſpoſed according to the Dimenſions of the Correlate Quantity, and then the Terms are to be multiply'd by any Arithmetical Progreſſion. But ſince neither this Rule, nor any other that I know yet publiſhed, extends to Equations affected with ſurd Quantities, without a previous Reduction; I ſhall give the following Example for that purpoſe.

4. EXAMP. 2. If the greateſt Quantity y in the Equation $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay+xx} = 0$ be to be determin'd, ſeek the Fluxions of x and y, and there will ariſe the Equation $3\dot{x}x^2 - 2a\dot{y}y + \frac{3ab\dot{y}y^2 + 2b\dot{y}y^3}{a^2+2ay+y^2} - \frac{4a\dot{x}xy + 6\dot{x}x^3 + a\dot{y}x^2}{2\sqrt{ay+xx}} = 0$. And ſince by ſuppoſition $\dot{y} = 0$, omit the Terms multiply'd by $\dot{y}$, (which, to ſhorten the labour, might have been done before, in the Operation,) and divide the reſt by $\dot{x}x$, and there will remain $3x - \frac{2ay+3xx}{\sqrt{ay+xx}} = 0$. When the Reduction is made, there will ariſe $4ay + 3xx = 0$, by help of which you may exterminate either of the quantities x or y out of the propos'd Equation, and then from the reſulting Equation, which will be Cubical, you may extract the Value of the other.

5. From this Problem may be had the Solution of theſe following.

I. *In a given Triangle, or in a Segment of any given Curve, to inſcribe the greateſt Rectangle.*

II. *To draw the greateſt or the leaſt right Line, which can lie between a given Point, and a Curve given in poſition. Or, to draw a Perpendicular to a Curve from a given Point.*

III. *To draw the greateſt or the leaſt right Lines, which paſſing through a given Point, can lie between two others, either right Lines or Curves.*

IV. *From a given Point within a Parabola, to draw a right Line, which ſhall cut the Parabola more obliquely than any other. And to do the ſame in other Curves.*

V. *To determine the Vertices of Curves, their greateſt or leaſt Breadths, the Points in which revolving parts cut each other,* &c.

VI. *To find the Points in Curves, where they have the greateſt or leaſt Curvature.*

VII. *To find the leaſt Angle in a given Ellipſis, in which the Ordinates can cut their Diameters.*

VIII. *Of Ellipſes that paſs through four given Points, to determine the greateſt, or that which approaches neareſt to a Circle.*

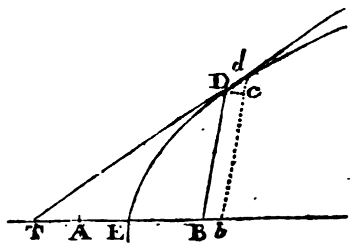
IX. *To determine ſuch a part of a Spherical Superficies, which can be illuminated, in its farther part, by Light coming from a great diſtance, and which is refracted by the nearer Hemiſphere.*

And many other Problems of a like nature may more eaſily be propoſed than reſolved, becauſe of the labour of Computation.

PROB. IV.

To draw Tangents to Curves.

Firſt Manner.



1. Tangents may be variouſly drawn, according to the various Relations of Curves to right Lines. And firſt let BD be a right Line, or Ordinate, in a given Angle to another right Line AB, as a Baſe or Abſciſs, and terminated at the Curve ED. Let this Ordinate move through an indefinitely ſmall Space to the place *bd*, ſo that it may be increaſed by the Moment *cd*, while AB is increaſed by the Moment B*b*, to which D*c* is equal and parallel. Let D*d* be produced till it meets with AB in T, and this Line will touch the Curve in D or *d*; and the Triangles *dc*D, DBT will be ſimilar. So that it is TB : BD :: D*c* (or B*b*) : *cd*.

2. Since therefore the Relation of BD to AB is exhibited by the Equation, by which the nature of the Curve is determined; ſeek for the Relation of the Fluxions, by Prob. 1. Then take TB to BD in the Ratio of the Fluxion of AB to the Fluxion of BD, and TD will touch the Curve in the Point D.

3. Ex. 1. Calling AB $= x$, and BD $= y$, let their Relation be $x^3 - ax^2 + axy - y^3 = 0$. And the Relation of the Fluxions will be $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + ay\dot{x} = 0$. So that $\dot{y} : \dot{x} :: 3xx - 2ax + ay : 3y^2 - ax ::$ BD (y) : BT. Therefore BT $= \frac{3y^3 - axy}{3x^2 - 2ax + ay}$. Therefore the Point D being given, and thence DB and AB, or y and x, the length BT will be given, by which the Tangent TD is determined.

4. But this Method of Operation may be thus concinnated. Make the Terms of the propoſed Equation equal to nothing: multiply by the proper number of the Dimenſions of the Ordinate, and put the Reſult in the Numerator: Then multiply the Terms of the ſame Equation by the proper number of the Dimenſions of the Abſciſs, and put the Product divided by the Abſciſs, in the Denominator of the Value of BT. Then take BT towards A, if its Value be affirmative, but the contrary way if that Value be negative.

5. Thus the Equation $\overset{0}{\underset{3}{x^3}} - \overset{0}{\underset{2}{ax^2}} + \overset{1}{\underset{1}{axy}} - \overset{3}{\underset{0}{y^3}} = 0$, being multiply'd by the upper Numbers, gives $axy - 3y^3$ for the Numerator; and multiply'd by the lower Numbers, and then divided by x, gives $3x^2 - 2ax + ay$ for the Denominator of the Value of BT.

6. Thus the Equation $y^3 - by^2 - cdy + bcd + dxy = 0$, (which denotes a Parabola of the ſecond kind, by help of which *Des Cartes* conſtructed Equations of ſix Dimenſions; ſee his Geometry, *p.* 42. *Amſterd. Ed. An.* 1659.) by Inſpection gives $\frac{3y^3 - 2by^2 - cdy + dxy}{dy}$, or $\frac{3yy}{d} - \frac{2by}{d} - c + x = $ BT.

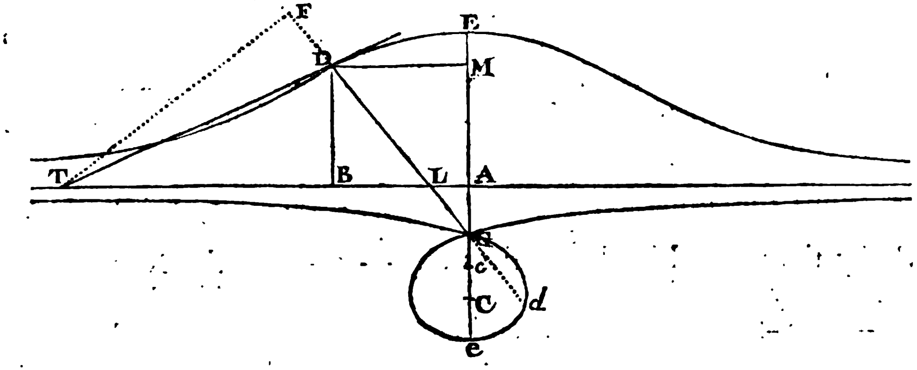
7. And thus $a^2 - \frac{r}{q}x^2 - y^2 = 0$, (which denotes an Ellipſis whoſe Center is A,) gives $\frac{-2yy}{-\frac{2r}{q}x}$, or $\frac{qyy}{rx} = $ BT. And ſo in others.

8. And you may take notice, that it matters not of what quantity the Angle of Ordination ABD may be.

9. But as this Rule does not extend to Equations affected by ſurd Quantities, or to mechanical Curves; in theſe Caſes we muſt have recourſe to the fundamental Method.

10. Ex. 2. Let $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay+xx} = 0$ be the Equation expreſſing the Relation between AB and BD; and by Prob. 1. the Relation of the Fluxions will be $3\dot{x}x^2 - 2a\dot{y}y + \frac{3ab\dot{y}y^2 + 2b\dot{y}y^3}{aa+2ay+yy} - \frac{4a\dot{x}xy - 6\dot{x}x^3 - a\dot{y}x^2}{2\sqrt{ay+xx}} = 0$. Therefore it will be $3xx - \frac{4axy - 6x^3}{2\sqrt{ay+xx}} : -2ay + \frac{3aby^2 + 2by^3}{aa+2ay+yy} + \frac{axx}{2\sqrt{ay+xx}} :: (\dot{y} : \dot{x} ::)$ BD : BT.

11. Ex. 3. Let ED be the Conchoid of *Nicomedes*, deſcribed with the Pole G, the Aſymptote AT, and the Diſtance LD; and let



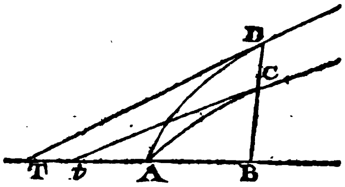
$GA = b$, $LD = c$, $AB = x$, and $BD = y$. And becauſe of ſimilar Triangles DBL and DMG, it will be LB : BD :: DM : MG; that is, $\sqrt{cc - yy} : y :: x : b + y$, and therefore $\overline{b+y}\sqrt{cc-yy} = yx$. Having got this Equation, I ſuppoſe $\sqrt{cc-yy} = z$, and thus I ſhall have two Equations $bz + yz = yx$, and $zz = cc - yy$. By the help of theſe I find the Fluxions of the Quantities $\dot{x}$, $\dot{y}$, and $\dot{z}$, by Prob. 1. From the firſt ariſes $b\dot{z} + \dot{y}z + y\dot{z} = \dot{y}x + \dot{x}y$, and from the ſecond $2\dot{z}z = -2\dot{y}y$, or $\dot{z}z + \dot{y}y = 0$. Out of theſe if we exterminate $\dot{z}$, there will ariſe $-\frac{b\dot{y}y}{z} - \frac{\dot{y}y^{2}}{z} + \dot{y}z = \dot{y}x + \dot{x}y$, which being reſolved it will be $y : z - \frac{by}{z} - \frac{yy}{z} - x :: (\dot{y} : \dot{x} ::)$ BD : BT. But as BD is y, therefore $\mathrm{BT} = z - x \frac{-by-yy}{z}$. That is, $-\mathrm{BT} = \mathrm{AL} + \frac{\mathrm{BD} \times \mathrm{GM}}{\mathrm{BL}}$; where the Sign $-$ prefixt to BT denotes, that the Point T muſt be taken contrary to the Point A.

12. SCHOLIUM. And hence it appears by the bye, how that point of the Conchoid may be found, which ſeparates the concave from the convex part. For when AT is the leaſt poſſible, D will be that point. Therefore make $\mathrm{AT} = v$; and ſince $\mathrm{BT} = -z + x + \frac{by + yy}{z}$, then $v = -z + 2x + \frac{by + yy}{z}$. Here to ſhorten the work, for x ſubſtitute $\frac{bz + yz}{y}$, which Value is derived from what is before, and it will be $\frac{2bz}{y} + z + \frac{by + yy}{z} = v$. Whence the Fluxions $\dot{v}$, $\dot{y}$, and $\dot{z}$ being found by Prob. 1. and ſuppoſing $\dot{v} = 0$,

by Prob. 3. there will ariſe $\frac{2b\dot{z}}{y}-\frac{2b\dot{y}z}{yy}+\dot{z}+\frac{b\dot{y}+2\dot{y}y}{z}-\frac{b\dot{z}y+\dot{z}yy}{zz}=\dot{v}=0$. Laſtly, ſubſtituting in this $\frac{-y\dot{y}}{z}$ for $\dot{z}$, and $cc-yy$ for zz, (which values of $\dot{z}$ and zz are had from what goes before,) and making a due Reduction, you will have $y^3+3by^2-2bc^2=0$. By the Conſtruction of which Equation y or AM, will be given. Then thro' M drawing MD parallel to AB, it will fall upon the Point D of contrary Flexure.

13. Now if the Curve be Mechanical whoſe Tangent is to be drawn, the Fluxions of the Quantities are to be found, as in Examp. 5. of Prob. 1. and then the reſt is to be perform'd as before.

14. Ex. 4. Let AC and AD be two Curves, which are cut in the Points C and D by the right Line BCD, apply'd to the Abſciſs AB in a given Angle. Let AB $=x$, BD $=y$, and $\frac{\text{Area ACB}}{1}=z$. Then (by Prob. 1. Preparat. to Examp. 5.) it will be $\dot{z}=\dot{x}\times$ BC.



15. Now let AC be a Circle, or any known Curve; and to determine the other Curve AD, let any Equation be propoſed, in which z is involved, as $zz+axz=y^4$. Then by Prob. 1. $2\dot{z}z+ax\dot{z}+a\dot{x}z=4\dot{y}y^3$. And writing $\dot{x}\times$ BC for $\dot{z}$, it will be $2\dot{x}z\times$ BC $+a\dot{x}x\times$ BC $+a\dot{x}z=4\dot{y}y^3$. Therefore $2z\times$ BC $+ax\times$ BC $+az:4y^3::(\dot{y}:\dot{x}::)$ BD : BT. So that if the nature of the Curve AC be given, the Ordinate BC, and the Area ACB or z; the Point T will be given, through which the Tangent DT will paſs.

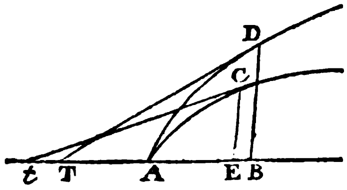
16. After the ſame manner, if $3z=2y$ be the Equation to the Curve AD; 'twill be $(3\dot{z})$ $3\dot{x}\times$ BC $=2\dot{y}$. So that 3BC : 2 :: $(\dot{y}:\dot{x}::)$ BD : BT. And ſo in others.

17. Ex. 5. Let AB $=x$, BD $=y$, as before, and let the length of any Curve AC be z. And drawing a Tangent to it, as Ct, 'twill be Bt : Ct :: $\dot{x}:\dot{z}$, or $\dot{z}=\frac{\dot{x}\times \text{C}t}{\text{B}t}$.

18. Now for determining the other Curve AD, whoſe Tangent is to be drawn, let there be given any Equation in which z is involved, ſuppoſe $z=y$. Then it will be $\dot{z}=\dot{y}$, ſo that Ct : Bt :: $(\dot{y}:\dot{x}::)$: BD : BT. But the Point T being found, the Tangent DT may be drawn.

19. Thus ſuppoſing $xz = yy$, 'twill be $\dot{x}z + \dot{z}x = 2\dot{y}y$; and for $\dot{z}$ writing $\frac{\dot{x}\times Ct}{Bt}$, there will ariſe $\dot{x}z + \frac{\dot{x}x \times Ct}{Bt} = 2\dot{y}y$. Therefore $z + \frac{x\times Ct}{Bt} : 2y :: BD : DT$.

20. Ex. 6. Let AC be a Circle, or any other known Curve, whoſe Tangent is C*t*, and let AD be any other Curve whoſe Tangent DT is to be drawn, and let it be defin'd by aſſuming AB = to the Arch AC; and (CE, BD being Ordinates to AB in a given Angle,) let the Relation of BD to CE or AE be expreſs'd by any Equation.



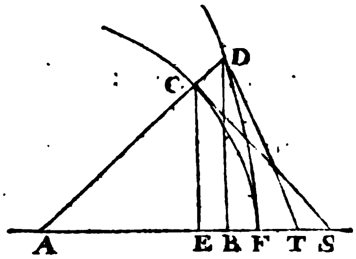
21. Therefore call AB or AC $= x$, BD $= y$, AE $= z$, and CE $= v$. And it is plain that $\dot{v}$, $\dot{x}$, and $\dot{z}$, the Fluxions of CE, AC, and AE, are to each other as CE, C*t*, and E*t*. Therefore $\dot{x} \times \frac{CE}{Ct} = \dot{v}$, and $\dot{x} \times \frac{Et}{Ct} = \dot{z}$.

22. Now let any Equation be given to define the Curve AD, as $y = z$. Then $\dot{y} = \dot{z}$; and therefore $Et : Ct :: (\dot{y} : \dot{x} ::)$ BD : BT.

23. Or let the Equation be $y = z + v - x$, and it will be $\dot{y} = (\dot{v} + \dot{z} - \dot{x} =) \dot{x} \times \frac{CE + Et - Ct}{Ct}$. And therefore $CE + Et - Ct : Ct :: (\dot{y} : \dot{x} ::)$ BD : BT.

24. Or finally, let the Equation be $ayy = v^3$, and it will be $2a\dot{y}y = (3\dot{v}v^2 =) 3\dot{x}v^2 \times \frac{CE}{Ct}$. So that $3v^2 \times CE : 2ay \times Ct ::$ BD : BT.

25. Ex. 7. Let FC be a Circle, which is touched by CS in C; and let FD be a Curve, which is defined by aſſuming any Relation of the Ordinate DB to the Arch FC, which is intercepted by DA drawn to the Center. Then letting fall CE, the Ordinate in the Circle, call AC or AF $= 1$, AB $= x$, DB $= y$, AE $= z$, CE $= v$, CF $= t$; and it will be $\dot{t}z = (\dot{t} \times \frac{CE}{CS} =)$ $\dot{v}$, and $-\dot{t}v = (\dot{t} \times \frac{-ES}{CS} =) \dot{z}$. Here I put $\dot{z}$ negatively, becauſe AE is diminiſh'd while EC is increaſed. And beſides AE : EC :: AB :

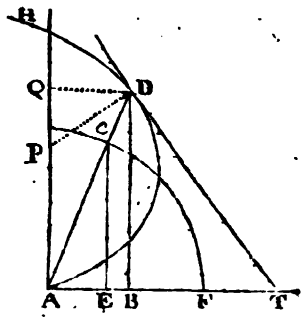


AB : BD, ſo that $zy = vx$, and thence by Prob. 1. $\dot{z}y + \dot{y}z = \dot{v}x + \dot{x}v$. Then exterminating $\dot{v}$, $\dot{z}$, and v, 'tis $\dot{y}x - \dot{t}y^2 - \dot{t}x^2 = \dot{x}y$.

26. Now let the Curve DF be defined by any Equation, from which the Value of $\dot{t}$ may be derived, to be ſubſtituted here. Suppoſe let $t = y$, (an Equation to the firſt Quadratrix,) and by Prob. 1. it will be $\dot{t} = \dot{y}$, ſo that $\dot{y}x - \dot{y}y^2 - \dot{y}x^2 = \dot{x}y$. Whence $y : xx + yy - x :: (\dot{y} : -\dot{x} ::)$ BD (y) : BT. Therefore BT $= x^2 + y^2 - x$; and AT $= xx + yy = \frac{ADq}{AF}$.

27. After the ſame manner, if it is $tt = by$, there will ariſe $2\dot{t}t = b\dot{y}$; and thence AT $= \frac{b}{2t} \times \frac{ADq}{AF}$. And ſo of others.

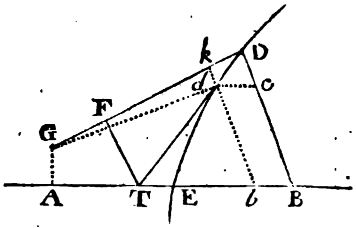
28. Ex. 8. Now if AD be taken equal to the Arch FC, the Curve ADH being then the Spiral of *Archimedes*; the ſame names of the Lines ſtill remaining as were put afore: Becauſe of the right Angle ABD 'tis $xx + yy = tt$, and therefore (by Prob. 1.) $\dot{x}x + \dot{y}y = \dot{t}t$. 'Tis alſo AD : AC :: DB : CE, ſo that $tv = y$, and thence (by Prob. 1.) $\dot{t}v + \dot{v}t = \dot{y}$. Laſtly, the Fluxion of the Arch FC is to the Fluxion of the right Line CE, as AC to AE, or as AD to AB, that is, $\dot{t} : \dot{v} :: t : x$, and thence $\dot{t}x = \dot{v}t$. Compare the Equations now found, and you will ſee 'tis $\dot{t}v + \dot{t}x = \dot{y}$, and thence $\dot{x}x + \dot{y}y = (\dot{t}t =) \frac{\dot{y}t}{v+x}$. And therefore compleating the Parallelogram ABDQ, if you make QD : QP :: (BD : BT :: $\dot{y} : -\dot{x}$ ::) $x : y - \frac{t}{v+x}$; that is, if you take AP $= \frac{t}{v+x}$, PD will be perpendicular to the Spiral.



29. And from hence (I imagine) it will be ſufficiently manifeſt, by what methods the Tangents of all ſorts of Curves are to be drawn. However it may not be foreign from the purpoſe, if I alſo ſhew how the Problem may be perform'd, when the Curves are refer'd to right Lines, after any other manner whatever: So that having the choice of ſeveral Methods, the eaſieſt and moſt ſimple may always be uſed.

Second Manner.

30. Let D be a point in the Curve, from which the Subtenſe DG is drawn to a given Point G, and let DB be an Ordinate in any given Angle to the Abſciſs AB. Now let the Point D flow for an infinitely ſmall ſpace D*d* in the Curve, and in GD let G*k* be taken equal to G*d*, and let the Parallelogram *dc*B*b* be compleated. Then D*k* and D*c* will be the contemporary Moments of GD and BD, by which they are diminiſh'd while D is transfer'd to *d*. Now let the right Line D*d* be produced, till it meets with AB in T, and from the Point T to the Subtenſe GD let fall the perpendicular TF, and then the Trapezia D*cdk* and DBTF will be like; and therefore DB : DF :: D*c* : D*k*.



31. Since then the Relation of BD to GD is exhibited by the Equation for determining the Curve; find the Relation of the Fluxions, and take FD to DB in the Ratio of the Fluxion of GD to the Fluxion of BD. Then from F raiſe the perpendicular FT, which may meet with AB in T, and DT being drawn will touch the Curve in D. But DT muſt be taken towards G, if it be affirmative, and the contrary way if negative.

32. Ex. 1. Call GD $= x$, and BD $= y$, and let their Relation be $x^3 - ax^2 + axy - y^3 = 0$. Then the Relation of the Fluxions will be $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$. Therefore $3xx - 2ax + ay : 3yy - ax :: (\dot{y} : \dot{x} ::)$ DB (y) : DF. So that DF $= \frac{3y^3 - axy}{3x^2 - 2ax + ay}$. Then any Point D in the Curve being given, and thence BD and GD or y and x, the Point F will be given alſo. From whence if the Perpendicular FT be raiſed, from its concourſe T with the Abſciſs AB, the Tangent DT may be drawn.

33. And hence it appears, that a Rule might be derived here, as well as in the former Caſe. For having diſpoſed all the Terms of the given Equation on one ſide, multiply by the Dimenſions of the Ordinate y, and place the reſult in the Numerator of a Fraction. Then multiply its Terms ſeverally by the Dimenſions of the Subtenſe x, and dividing the reſult by that Subtenſe x, place the Quotient in the Denominator of the Value of DF. And take the ſame Line DF towards G if it be affirmative, otherwiſe the contrary way. Where you

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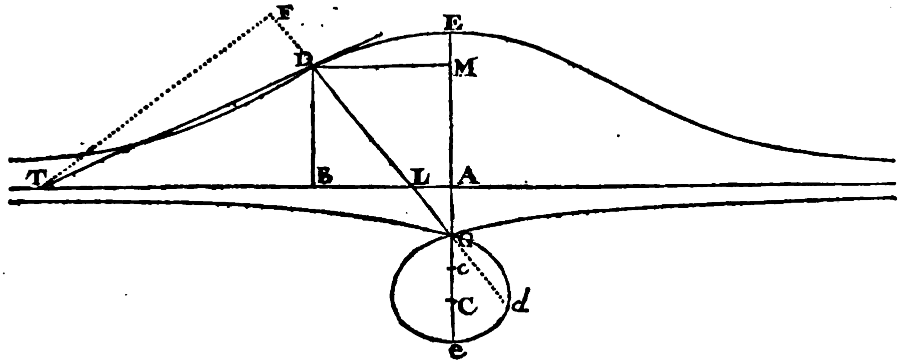
you may obſerve, that it is no matter how far diſtant the Point G is from the Abſciſs AB, or if it be at all diſtant, nor what is the Angle of Ordination ABD.

34. Let the Equation be as before $x^3 - ax^2 + axy - y^3 = 0$; it gives immediately $axy - 3y^3$ for the Numerator, and $3x^2 - 2ax + ay$ for the Denominator of the Value of DF.

35. Let alſo $a + \frac{b}{a}x - y = 0$, (which Equation is to a Conick Section,) it gives $-y$ for the Numerator, and $\frac{b}{a}$ for the Denominator of the Value of DF, which therefore will be $-\frac{ay}{b}$.

36. And thus in the Conchoid, (wherein theſe things will be perform'd more expeditiouſly than before,) putting $GA = b$, $LD = c$, $GD = x$, and $BD = y$, it will be BD (y) : DL (c) :: GA (b) : GL ($x - c$). Therefore $xy - cy = cb$, or $xy - cy - cb = 0$. This Equation according to the Rule gives $\frac{xy - cy}{y}$, that is, $x - c = \text{DF}$. Therefore prolong GD to F, ſo that DF = LG, and at F raiſe the perpendicular FT meeting the Aſymptote AB in T, and DT being drawn will touch the Conchoid.



37. But when compound or ſurd Quantities are found in the Equation, you muſt have recourſe to the general Method, except you ſhould chuſe rather to reduce the Equation.

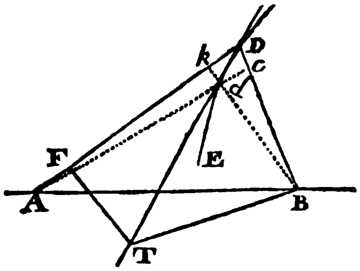
38. Ex. 2. If the Equation $\overline{b+y} \times \sqrt{cc - yy} = yx$, were given for the Relation between GD and BD; (ſee the foregoing Figure, p. 52.) find the Relation of the Fluxions by Prob. 1. As ſuppoſing $\sqrt{cc - yy} = z$, you will have the Equations $bz + yz = yx$, and $cc - yy = zz$, and thence the Relation of the Fluxions $b\dot{z} + y\dot{z}$ $\dot{y}z = \dot{y}x + y\dot{x}$, and $-2\dot{y}y = 2\dot{z}z$. And now $\dot{z}$ and z being

exter-

exterminated, there will ariſe $\dot{y}\sqrt{cc-yy}-\frac{b\dot{y}y+\dot{y}y^2}{\sqrt{cc-yy}}-\dot{y}x=\dot{x}y$. Therefore $y:\sqrt{cc-yy}-\frac{by+yy}{\sqrt{cc-yy}}-x::(\dot{y}:\dot{x}::)$ BD (y) : DF.

Third Manner.

39. Moreover, if the Curve be refer'd to two Subtenſes AD and BD, which being drawn from two given Points A and B, may meet at the Curve: Conceive that Point D to flow on through an infinitely little Space D*d* in the Curve; and in AD and BD take A*k* = A*d*, and B*c* = B*d*; and then *k*D and *c*D will be contemporaneous Moments of the Lines AD and BD. Take therefore DF to BD in the Ratio of the Moment D*k* to the Moment D*c*, (that is, in the Ratio of the Fluxion of the Line AD to the Fluxion of the Line BD,) and draw BT, FT perpendicular to BD, AD, meeting in T. Then the Trapezia DFTB and D*kdc* will be ſimilar, and therefore the Diagonal DT will touch the Curve.

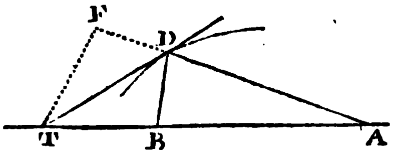


40. Therefore from the Equation, by which the Relation is defined between AD and BD, find the Relation of the Fluxions by Prob. I. and take FD to BD in the ſame Ratio.

41. EXAMP. Suppoſing AD $=x$, and BD $=y$, let their Relation be $a+\frac{ex}{d}-y=0$. This Equation is to the Ellipſes of the ſecond Order, whoſe Properties for Refracting of Light are ſhewn by *Des Cartes*, in the ſecond Book of his Geometry. Then the Relation of the Fluxions will be $\frac{e\dot{x}}{d}-\dot{y}=0$. 'Tis therefore $e:d::(\dot{y}:\dot{x}::)$ BD : DF.

42. And for the ſame reaſon if $a-\frac{ex}{d}-y=0$, 'twill be $e:-d::$ BD : DF. In the firſt Caſe take DF towards A, and contrary-wiſe in the other caſe.

43. COROL. I. Hence if $d=e$, (in which caſe the Curve becomes a Conick Section,) 'twill be DF = DB. And therefore the Triangles DFT and DBT being equal, the Angle FDB will be biſected by the Tangent.

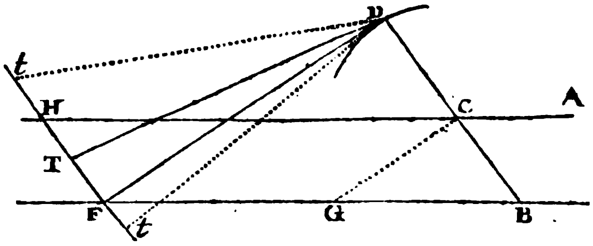


44. COROL. 2. And hence alſo thoſe things will be manifeſt of themſelves, which are demonſtrated, in a very prolix manner, by *Des Cartes* concerning the Refraction of theſe Curves. For as much as DF and DB, (which are in the given Ratio of *d* to *e*,) in reſpect of the Radius DT, are the Sines of the Angles DTF and DTB, that is, of the Ray of Incidence AD upon the Surface of the Curve, and of its Reflexion or Refraction DB. And there is a like reaſoning concerning the Refractions of the Conick Sections, ſuppoſing that either of the Points A or B be conceived to be at an infinite diſtance.

45. It would be eaſy to modify this Rule in the manner of the foregoing, and to give more Examples of it: As alſo when Curves are refer'd to Right lines after any other manner, and cannot commodiouſly be reduced to the foregoing, it will be very eaſy to find out other Methods in imitation of theſe, as occaſion ſhall require.

Fourth Manner.

46. As if the right Line BCD ſhould revolve about a given Point B, and one of its Points D ſhould deſcribe a Curve, and another Point C ſhould be the interſection of the right Line BCD, with another right Line AC given in poſition. Then the Relation of BC and BD being expreſs'd by any Equation; draw BF parallel to AC, ſo as to meet DF, perpendicular to BD, in F. Alſo erect FT perpendicular to DF; and take FT in the ſame Ratio to BC, that the Fluxion of BD has to the Fluxion of BC. Then DT being drawn will touch the Curve.



Fifth Manner.

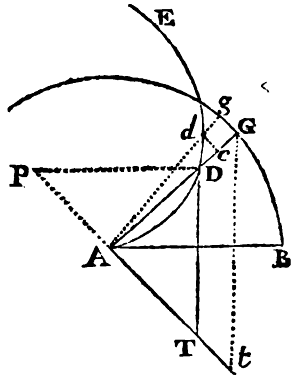
47. But if the Point A being given, the Equation ſhould expreſs the Relation between AC and BD; draw CG parallel to DF, and take FT in the ſame Ratio to BG, that the Fluxion of BD has to the Fluxion of AC.

Sixth Manner.

48. Or again, if the Equation expreſſes the Relation between AC and CD; let AC and FT meet in H; and take HT in the ſame Ratio to BG, that the Fluxion of CD has to the Fluxion of AC. And the like in others.

Seventh Manner: For Spirals.

49. The Problem is not otherwiſe perform'd, when the Curves are refer'd, not to right Lines, but to other Curve-lines, as is uſual in Mechanick Curves. Let BG be the Circumference of a Circle, in whoſe Semidiameter AG, while it revolves about the Center A, let the Point D be conceived to move any how, ſo as to deſcribe the Spiral ADE. And ſuppoſe D*d* to be an infinitely little part of the Curve thro' which D flows, and in AD take A*c* = A*d*, then *c*D and G*g* will be contemporaneous Moments of the right Line AD and of the Periphery BG. Therefore draw A*t* parallel to *cd*, that is, perpendicular to AD, and let the Tangent DT meet it in T; then it will be *c*D : *cd* :: AD : AT. Alſo let G*t* be parallel to the Tangent DT, and it will be *cd* : G*g* :: (A*d* or AD : AG ::) AT : A*t*.



50. Therefore any Equation being propoſed, by which the Relation is expreſs'd between BG and AD; find the Relation of their Fluxions by Prob. 1. and take A*t* in the ſame Ratio to AD: And then G*t* will be parallel to the Tangent.

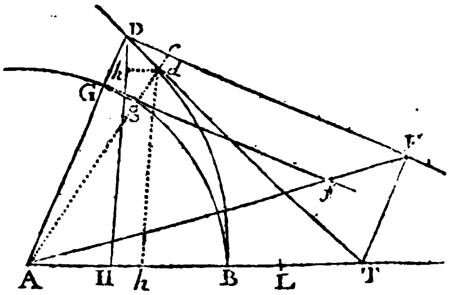
51. Ex. 1. Calling BG = x, and AD = y, let their Relation be $x^3 - ax^2 + axy - y^3 = 0$, and by Prob. 1. $3x^2 - 2ax + ay : 3y^2 - ax :: (\dot{y} : \dot{x} ::)$ AD : A*t*. The Point *t* being thus found, draw G*t*, and DT parallel to it, which will touch the Curve.

52. Ex. 2. If 'tis $\frac{ax}{b} = y$, (which is the Equation to the Spiral of *Archimedes*,) 'twill be $\frac{a\dot{x}}{b} = \dot{y}$, and therefore $a : b :: (\dot{y} : \dot{x} ::)$ AD : A*t*. Wherefore by the way, if TA be produced to P, that it may be AP : AB :: $a : b$, PD will be perpendicular to the Curve.

53. Ex. 3. If $xx = by$, then $2\dot{x}x = b\dot{y}$, and $2x : b ::$ AD : A*t*. And thus Tangents may be eaſily drawn to any Spirals whatever.

Eighth Manner : For Quadratrices.

54. Now if the Curve be ſuch, that any Line AGD, being drawn from the Center A, may meet the Circular Arch in G, and the Curve in D; and if the Relation between the Arch BG, and the right Line DH, which is an Ordinate to the Baſe or Abſciſs AH in a given Angle, be determin'd by any Equation whatever: Conceive the Point D to move in the Curve for an infinitely ſmall Interval to *d*, and the Parallelogram *dh*H*k* being compleated, produce A*d* to *c*, ſo that A*c* = AD; then G*g* and D*k* will be contemporaneous Moments of the Arch BG and of the Ordinate DH. Now produce D*d* ſtrait on to T, where it may meet with AB, and from thence let fall the Perpendicular TF on D*c*F. Then the Trapezia D*kdc* and DHTF will be ſimilar; and therefore D*k* : D*c* :: DH : DF. And beſides if G*f* be raiſed perpendicular to AG, and meets AF in *f*; becauſe of the Parallels DF and G*f*, it will be D*c* : G*g* :: DF : G*f*. Therefore *ex æquo*, 'tis D*k* : G*g* :: DH : G*f*, that is, as the Moments or Fluxions of the Lines DH and BG.



55. Therefore by the Equation which expreſſes the Relation of BG to DH, find the Relation of the Fluxions (by Prob. 1.) and in that Ratio take G*f*, the Tangent of the Circle BG, to DH. Draw DF parallel to G*f*, which may meet A*f* produced in F. And at F erect the perpendicular FT, meeting AB in T; and the right Line DT being drawn, will touch the Quadratrix.

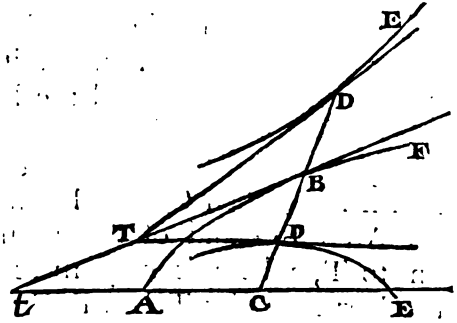
56. Ex. 1. Making BG $= x$, and DH $= y$, let it be $xx = by$; then (by Prob. 1.) $2x\dot{x} = b\dot{y}$. Therefore $2x : b :: (\dot{y} : \dot{x} ::)$ DH : G*f*; and the Point *f* being found, the reſt will be determin'd as above.

But perhaps this Rule may be thus made ſomething neater: Make $\dot{x} : \dot{y} ::$ AB : AL. Then AL : AD :: AD : AT, and then DT will touch the Curve. For becauſe of equal Triangles AFD and ATD, 'tis AD × DF = AT × DH, and therefore AT : AD :: (DF or $\frac{AD}{AG} \times Gf : DH$ or $\frac{\dot{y}}{\dot{x}} Gf ::)$ AD : $\left(\frac{\dot{y}}{\dot{x}}\right.$ AG or) AL.

57. Ex. 2. Let $x = y$, (which is the Equation to the Quadratrix of the Ancients,) then $\dot{x} = \dot{y}$. Therefore AB : AD :: AD : AT.

58. Ex. 3. Let $axx = y^3$, then $2a\dot{x}x = 3\dot{y}y^2$. Therefore make $3y^2 : 2ax :: (\dot{x} : \dot{y} ::)$ AB : AL. Then AL : AD :: AD : AT. And thus you may determine expeditiouſly the Tangents of any other Quadratrices, howſoever compounded.

Ninth Manner.



59. Laſtly, if ABF be any given Curve, which is touch'd by the right Line Bt; and a part BD of the right Line BC, (being an Ordinate in any given Angle to the Abſciſs AC,) intercepted between this and another Curve DE, has a Relation to the portion of the Curve AB, which is expreſs'd by any Equation: You may draw a Tangent DT to the other Curve, by taking (in the Tangent of this Curve,) BT in the ſame Ratio to BD, as the Fluxion of the Curve AB hath to the Fluxion of the right Line BD.

60. Ex. 1. Calling AB $= x$, and BD $= y$; let it be $ax = yy$, and therefore $a\dot{x} = 2\dot{y}y$. Then $a ; 2y :: (\dot{y} : \dot{x} ::)$ BD : BT.

61. Ex. 2. Let $\frac{a}{b}x = y$, (the Equation to the Trochoid, if ABF be a Circle,) then $\frac{a}{b}\dot{x} = \dot{y}$, and $a : b ::$ BD : BT.

62. And with the ſame eaſe may Tangents be drawn, when the Relation of BD to AC, or to BC, is expreſs'd by any Equation; or when the Curves are refer'd to right Lines, or to any other Curves, after any other manner whatever.

63. There are alſo many other Problems, whoſe Solutions are to be derived from the ſame Principles; ſuch as theſe following.

I. *To find a Point of a Curve, where the Tangent is parallel to the Abſciſs, or to any other right Line given in poſition; or is perpendicular to it, or inclined to it in any given Angle.*

II. *To find the Point where the Tangent is moſt or leaſt inclined to the Abſciſs, or to any other right Line given in poſition. That is, to find the confine of contrary Flexure.* Of this I have already given a Specimen in the Conchoid.

III. *From any given Point without the Perimeter of a Curve, to draw a right Line, which with the Perimeter may make an Angle of Contact,*

Contact, or a right Angle, or any other given Angle. That is, from a given Point, to draw Tangents, or Perpendiculars, or right Lines that ſhall have any other Inclination to a Curve-line.

IV. *From any given Point within a Parabola, to draw a right Line, which may make with the Perimeter the greateſt or leaſt Angle poſſible. And ſo of all Curves whatever.*

V. *To draw a right Line which may touch two Curves given in poſition, or the ſame Curve in two Points, when that can be done.*

VI. *To draw any Curve with given Conditions, which may touch another Curve given in poſition, in a given Point.*

VII. *To determine the Refraction of any Ray of Light, that falls upon any Curve Superficies.*

The Reſolution of theſe, or of any other the like Problems, will not be ſo difficult, abating the tedioųſneſs of Computation, as that there is any occaſion to dwell upon them here: And I imagine it may be more agreeable to Geometricians barely to have mention'd them.

PROB. V.

At any given Point of a given Curve, to find the Quantity of Curvature.

1. There are few Problems concerning Curves more elegant than this, or that give a greater Inſight into their nature. In order to its Reſolution, I muſt premiſe theſe following general Conſiderations.

2. I. The ſame Circle has every where the ſame Curvature, and in different Circles it is reciprocally proportional to their Diameters. If the Diameter of any Circle is as little again as the Diameter of another, the Curvature of its Periphery will be as great again. If the Diameter be one-third of the other, the Curvature will be thrice as much, *&c.*

3. II. If a Circle touches any Curve on its concave ſide, in any given Point, and if it be of ſuch magnitude, that no other tangent Circle can be interſcribed in the Angles of Contact near that Point; that Circle will be of the ſame Curvature as the Curve is of, in that Point of Contact. For the Circle that comes between the Curve and another Circle at the Point of Contact, varies leſs from the Curve, and makes a nearer approach to its Curvature, than that other Circle does. And therefore that Circle approaches neareſt to its

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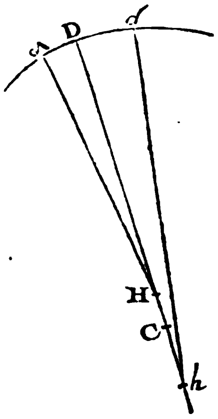
Curvature, between which and the Curve no other Circle can intervene.

4. III. Therefore the Center of Curvature to any Point of a Curve, is the Center of a Circle equally curved. And thus the Radius or Semidiameter of Curvature is part of the Perpendicular to the Curve, which is terminated at that Center.

5. IV. And the proportion of Curvature at different Points will be known from the proportion of Curvature of æqui-curve Circles, or from the reciprocal proportion of the Radii of Curvature.

6. Therefore the Problem is reduced to this, that the Radius, or Center of Curvature may be found.

7. Imagine therefore that at three Points of the Curve δ, D, and *d*, Perpendiculars are drawn, of which thoſe that are at D and δ meet in H, and thoſe that are at D and *d* meet in *h*: And the Point D being in the middle, if there is a greater Curvity at the part Dδ than at D*d*, then DH will be leſs than *dh*. But by how much the Perpendiculars δH and *dh* are nearer the intermediate Perpendicular, ſo much the leſs will the diſtance be of the Points H and *h*: And at laſt when the Perpendiculars meet, thoſe Points will coincide. Let them coincide in the Point C, then will C be the Center of Curvature, at the Point D of the Curve, on which the Perpendiculars ſtand; which is manifeſt of itſelf.



8. But there are ſeveral Symptoms or Properties of this Point C, which may be of uſe to its determination.

9. I. That it is the Concourſe of Perpendiculars that are on each ſide at an infinitely little diſtance from DC.

10. II. That the Interſections of Perpendiculars, at any little finite diſtance on each ſide, are ſeparated and divided by it; ſo that thoſe which are on the more curved ſide Dδ ſooner meet at H, and thoſe which are on the other leſs curved ſide D*d* meet more remotely at *h*.

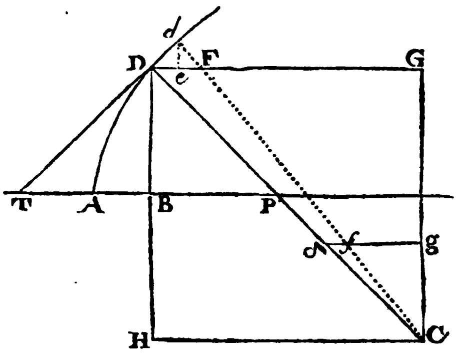
11. III. If DC be conceived to move, while it inſiſts perpendicularly on the Curve, that point of it C, (if you except the motion of approaching to or receding from the Point of Inſiſtence C,) will be leaſt moved, but will be as it were the Center of Motion.

12. IV. If a Circle be deſcribed with the Center C, and the diſtance DC, no other Circle can be deſcribed, that can lie between at the Contact.

13. V. Laſtly, if the Center H or *h* of any other touching Circle approaches by degrees to C the Center of this, till at laſt it coincides with it; then any of the points in which that Circle ſhall cut the Curve, will coincide with the point of Contact D.

14. And each of theſe Properties may ſupply the means of ſolving the Problem different ways: But we ſhall here make choice of the firſt, as being the moſt ſimple.

15. At any Point D of the Curve let DT be a Tangent, DC a Perpendicular, and C the Center of Curvature, as before. And let AB be the Abſciſs, to which let DB be apply'd at right Angles, and which DC meets in P. Draw DG parallel to AB, and CG perpendicular to it, in which take C*g* of any given Magnitude, and draw *g*δ perpendicular to it, which meets DC in δ. Then it will be C*g* : *g*δ :: (TB : BD ::) the Fluxion of the Abſciſs, to the Fluxion of the Ordinate. Likewiſe imagine the Point D to move in the Curve an infinitely little diſtance D*d*, and drawing *de* perpendicular to DG, and C*d* perpendicular to the Curve, let C*d* meet DG in F, and δ*g* in *f*. Then will D*e* be the Momentum of the Abſciſs, *de* the Momentum of the Ordinate, and δ*f* the contemporaneous Momentum of the right Line *g*δ. Therefore DF $= De + \frac{de \times de}{De}$. Having therefore the Ratio's of theſe Moments, or, which is the ſame thing, of their generating Fluxions, you will have the Ratio of CG to the given Line C*g*, (which is the ſame as that of DF to δ*f*,) and thence the Point C will be determined.



16. Therefore let AB $= x$, BD $= y$, C*g* $= 1$, and *g*δ $= z$; then it will be $1 : z :: \dot{x} : \dot{y}$, or $z = \frac{\dot{y}}{\dot{x}}$. Now let the Momentum δ*f* of z be $\dot{z} \times o$, (that is, the Product of the Velocity and of an infinitely ſmall Quantity o,) and therefore the Moments D*e* $= \dot{x} \times o$, *de* $= \dot{y} \times o$, and thence DF $= \dot{x}o + \frac{\dot{y}\dot{y}o}{\dot{x}}$. Therefore 'tis C*g* (1) : CG :: (δ*f* : DF ::) $\dot{z}o : \dot{x}o + \frac{\dot{y}\dot{y}o}{\dot{x}}$. That is, CG $= \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{\dot{x}\dot{z}}$.

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17. And whereas we are at liberty to aſcribe whatever Velocity we pleaſe to the Fluxion of the Abſciſs $\dot{x}$, (to which, as to an equable Fluxion, the reſt may be referr'd;) make $\dot{x}=1$, and then $\dot{y}=z$, and $CG=\frac{1+zz}{\dot{z}}$. And thence $DG=\frac{z+z^3}{\dot{z}}$, and $DC=\frac{\overline{1+zz}\sqrt{1+zz}}{\dot{z}}$.

18. Therefore any Equation being propoſed, in which the Relation of BD to AB is expreſs'd for defining the Curve; firſt find the Relation betwixt $\dot{x}$ and $\dot{y}$, by Prob. 1. and at the ſame time ſubſtitute 1 for $\dot{x}$, and z for $\dot{y}$. Then from the Equation that ariſes, by the ſame Prob. 1. find the Relation between $\dot{x}$, $\dot{y}$, and $\dot{z}$, and at the ſame time ſubſtitute 1 for $\dot{x}$, and z for $\dot{y}$, as before. And thus by the former operation you will obtain the Value of z, and by the latter you will have the Value of $\dot{z}$; which being obtain'd, produce DB to H, towards the concave part of the Curve, that it may be $DH=\frac{1+zz}{\dot{z}}$, and draw HC parallel to AB, and meeting the Perpendicular DC in C; then will C be the Center of Curvature at the Point D of the Curve. Or ſince it is $1+zz=\frac{PT}{BT}$, make $DH=\frac{PT}{\dot{z}\times BT}$, or $DC=\frac{\overline{DP}|^3}{\dot{z}\times\overline{DB}|^3}$.

19. Ex. 1. Thus the Equation $ax+bx^2-y^2=0$ being propoſed, (which is an Equation to the Hyperbola whoſe Latus rectum is a, and Tranſverſum $\frac{a}{b}$;) there will ariſe (by Prob. 1.) $a+2bx-2zy=0$, (writing 1 for $\dot{x}$, and z for $\dot{y}$ in the reſulting Equation, which otherwiſe would have been $a\dot{x}+2b\dot{x}x-2\dot{y}y=0$;) and hence again there ariſes $2b-2zz-2\dot{z}y=0$, (1 and z being again wrote for $\dot{x}$ and $\dot{y}$.) By the firſt we have $z=\frac{a+2bx}{2y}$, and by the latter $\dot{z}=\frac{b-zz}{y}$. Therefore any Point D of the Curve being given, and conſequently x and y, from thence z and $\dot{z}$ will be given, which being known, make $\frac{1+zz}{\dot{z}}=GC$ or DH, and draw HC.

20. As if definitely you make $a=3$, and $b=1$, ſo that $3x+xx=yy$ may be the condition of the Hyperbola. And if you aſſume $x=1$, then $y=2$, $z=\frac{5}{4}$, $\dot{z}=-\frac{9}{32}$, and $DH=-9\frac{1}{9}$. H being found, raiſe the Perpendicular HC meeting the Perpendicular

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cular DC before drawn; or, which is the ſame thing, make HD : HC :: (1 : z ::) 1 : $\frac{5}{4}$. Then draw DC the Radius of Curvature.

21. When you think the Computation will not be too perplex, you may ſubſtitute the indefinite Values of $\dot{z}$ and z into $\frac{1+zz}{\dot{z}}$, the Value of CG. Thus in the preſent Example, by a due Reduction you will have DH $= y + \frac{4y^3 + 4by^3}{aa}$. Yet the Value of DH by Calculation comes out negative, as may be ſeen in the numeral Example. But this only ſhews, that DH muſt be taken towards B; for if it had come out affirmative, it ought to have been drawn the contrary way.

22. COROL. Hence let the Sign prefixt to the Symbol $+b$ be changed, that it may be $ax - bxx - yy = 0$, (an Equation to the Ellipſis,) then DH $= y + \frac{4y^3 - 4by^3}{aa}$.

23. But ſuppoſing $b = 0$, that the Equation may become $ax - yy = 0$, (an Equation to the Parabola,) then DH $= y + \frac{4y^3}{aa}$; and thence DG $= \frac{1}{2}a + 2x$.

24. From theſe ſeveral Expreſſions it may eaſily be concluded, that the Radius of Curvature of any Conick Section, is always $\frac{4\,\overline{\mathrm{DP}}|^{3}}{aa}$.

25. Ex. 2. If $x^3 = ay^2 - xy^2$ be propoſed, (which is the Equation to the Ciſſoid of *Diocles*,) by Prob. 1. it will be firſt $3x^2 = 2azy - 2xzy - y^2$; and then $6x = 2a\dot{z}y + 2azz - 2zy - 2x\dot{z}y - 2xzz - 2zy$: So that $z = \frac{3xx + yy}{2ay - 2xy}$, and $\dot{z} = \frac{3x - azz + 2zy + xzz}{ay - xy}$. Therefore any Point of the Ciſſoid being given, and thence x and y, there will be given alſo z and $\dot{z}$; which being known, make $\frac{1+zz}{\dot{z}} =$ CG.

26. Ex. 3. If $\overline{b+y}\sqrt{cc-yy} = xy$ were given, (which is the Equation to the Conchoid, in pag. 48;) make $\sqrt{cc-yy} = v$, and there will ariſe $bv + yv = xy$. Now the firſt of theſe, $(cc - yy = vv,)$ will give (by Prob. 1.) $-2yz = 2\dot{v}v$, (writing z for $\dot{y}$;) and the latter will give $b\dot{v} + y\dot{v} + zv = y + xz$. And from theſe Equations rightly diſpoſed $\dot{v}$ and z will be determined. But that $\dot{z}$ may alſo be found; out of the laſt Equation exterminate the Fluxion $\dot{v}$, by ſubſtituting $-\frac{yz}{v}$, and there will ariſe $-\frac{byz}{v} - \frac{yyz}{v} + zv$ $= y$

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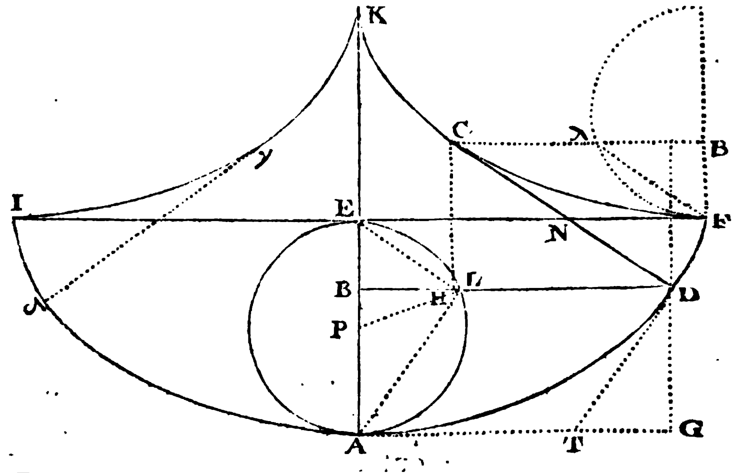
$= y + xz$, an Equation that comprehends the flowing Quantities, without any of their Fluxions, as the Reſolution of the firſt Problem requires. Hence therefore by Prob. 1. we ſhall have $-\frac{bz^2}{v} - \frac{b\dot{y}z}{v} + \frac{byz\dot{v}}{vv} - \frac{2yz\dot{z}}{v} - \frac{y\dot{y}z}{v} + \frac{yyz\dot{v}}{vv} + \dot{z}v + z\dot{v} = 2z + x\dot{z}$. This Equation being reduced, and diſpoſed in order, will give $\dot{z}$. But when z and $\dot{z}$ are known, make $\frac{1+zz}{\dot{z}} =$ CG.

27. If we had divided the laſt Equation but one by $\dot{z}$, then by Prob. 1. we ſhould have had $-\frac{bz}{v} + \frac{by\dot{v}}{vv} - \frac{2yz}{v} + \frac{yy\dot{v}}{vv} + \dot{v} = 2 - \frac{\dot{y}z}{zz}$; which would have been a more ſimple Equation than the former, for determining $\dot{z}$.

28. I have given this Example, that it may appear, how the operation is to be perform'd in ſurd Equations: But the Curvature of the Conchoid may be thus found a ſhorter way. The parts of the Equation $\overline{b+y}\sqrt{cc-yy} = xy$ being ſquared, and divided by yy, there ariſes $\frac{b^2c^2}{y^2} + \frac{2bc^2}{y} \begin{matrix} +c^2 \\ -b^2 \end{matrix} - 2by - y^2 = x^2$, and thence by Prob. 1. $-\frac{2b^2c^2z}{y^3} - \frac{2bc^2z}{y^2} - 2bz - 2yz = 2x$, or $-\frac{b^2c^2}{y^3} - \frac{bc^2}{y^2} - b - y = \frac{x}{z}$. And hence again by Prob. 1. $\frac{3b^2c^2z}{y^4} + \frac{2bc^2z}{y^3} - \dot{z} = \frac{1}{z} - \frac{x\dot{z}}{zz}$. By the firſt reſult z is determined, and $\dot{z}$ by the latter.

29. Ex. 4. Let ADF be a Trochoid [or Cycloid] belonging to the Circle ALE, whoſe Diameter is AE; and making the Ordinate BD to cut the Circle in L, call AE $= a$, AB $= x$, BD $= y$, BL $= v$, and the Arch AL $= t$, and the Fluxion of the ſame Arch $= \dot{t}$. And firſt (drawing the Semidiameter PL,) the Fluxion of the



Baſe or Abſciſs AB will be to the Fluxion of the Arch AL, as BL to PL; that is, $\dot{x}$ or $1 : \dot{t} :: v : \frac{1}{2}a$. And therefore $\frac{a}{2v} = \dot{t}$. Then from the nature of the Circle $ax - xx = vv$, and therefore by Prob. 1. $a - 2x = 2\dot{v}v$, or $\frac{a-2x}{2v} = \dot{v}$.

30. Moreover from the nature of the Trochoid, 'tis LD = Arch AL, and therefore $v + t = y$. And thence (by Prob. 1.) $\dot{v} + \dot{t} = z$. Laſtly, inſtead of the Fluxions $\dot{v}$ and $\dot{t}$ let their Values be ſubſtituted, and there will ariſe $\frac{a-x}{v} = z$. Whence (by Prob. 1.) is derived $-\frac{a\dot{v}}{vv} + \frac{x\dot{v}}{vv} - \frac{1}{v} = \dot{z}$. And theſe being found, make $\frac{1+zz}{\dot{z}} = -$ DH, and raiſe the perpendicular HC.

31. COR. 1. Now it follows from hence, that DH = 2BL, and CH = 2BE, or that EF biſects the radius of Curvature CD in N. And this will appear by ſubſtituting the values of $\dot{z}$ and z now found, in the Equation $\frac{1+zz}{\dot{z}} =$ DH, and by a proper reduction of the reſult.

32. COR. 2. Hence the Curve FCK, deſcribed indefinitely by the Center of Curvature of ADF, is another Trochoid equal to this, whoſe Vertices at I and F adjoin to the Cuſpids of this. For let the Circle Fλ, equal and alike poſited to ALE, be deſcribed, and let Cβ be drawn parallel to EF, meeting the Circle in λ: Then will Arch Fλ = (Arch EL = NF =) Cλ.

33. COR. 3. The right Line CD, which is at right Angles to the Trochoid IAF, will touch the Trochoid IKF in the point C.

34. COR. 4. Hence (in the inverted Trochoids,) if at the Cuſpid K of the upper Trochoid, a Weight be hung by a Thread at the diſtance KA or 2EA, and while the Weight vibrates, the Thread be ſuppos'd to apply itſelf to the parts of the Trochoid KF and KI, which reſiſt it on each ſide, that it may not be extended into a right Line, but compel it (as it departs from the Perpendicular) to be by degrees inflected above, into the Figure of the Trochoid, while the lower part CD, from the loweſt Point of Contact, ſtill remains a right Line: The Weight will move in the Perimeter of the lower Trochoid, becauſe the Thread CD will always be perpendicular to it.

35. COR. 5. Therefore the whole Length of the Thread KA is equal to the Perimeter of the Trochoid KCF, and its part CD is equal to the part of the Perimeter CF.

36. Cor. 6. Since the Thread by its oſcillating Motion revolves about the moveable Point C, as a Center; the Superficies through which the whole Line CD continually paſſes, will be to the Superficies through which the part CN above the right Line IF paſſes at the ſame time, as $\overline{CD}^2$ to $\overline{CN}^2$, that is, as 4 to 1. Therefore the Area CFN is a fourth part of the Area CFD; and the Area KCNE is a fourth part of the Area AKCD.

37. Cor. 7. Alſo ſince the ſubtenſe EL is equal and parallel to CN, and is converted about the immoveable Center E, juſt as CN moves about the moveable Center C; the Superficies will be equal through which they paſs in the ſame time, that is, the Area CFN, and the Segment of the Circle EL. And thence the Area NFD will be the triple of that Segment, and the whole area EADF will be the triple of the Semicircle.

38. Cor. 8. When the Weight D arrives at the point F, the whole Thread will be wound about the Perimeter of the Trochoid KCF, and the Radius of Curvature will there be nothing. Wherefore the Trochoid IAF is more curved, at its Cuſpid F, than any Circle; and makes an Angle of Contact, with the Tangent βF produced, infinitely greater than a Circle can make with a right Line.

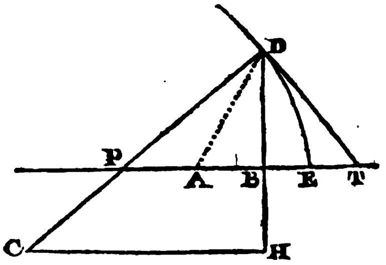
39. But there are Angles of Contact that are infinitely greater than Trochoidal ones, and others infinitely greater than theſe, and ſo on *in infinitum*; and yet the greateſt of them all are infinitely leſs than right-lined Angles. Thus $xx = ay$, $x^3 = by^2$, $x^4 = cy^3$, $x^5 = dy^4$, &c. denote a Series of Curves, of which every ſucceeding one makes an Angle of Contact with its Abſciſs, which is infinitely greater than the preceding can make with the ſame Abſciſs. And the Angle of Contact which the firſt $xx = ay$ makes, is of the ſame kind with Circular ones; and that which the ſecond $x^3 = by^2$ makes, is of the ſame kind with Trochoidals. And tho' the Angles of the ſucceeding Curves do always infinitely exceed the Angles of the preceding, yet they can never arrive at the magnitude of a right-lined Angle.

40. After the ſame manner $x = y$, $xx = ay$, $x^3 = b^2y$, $x^4 = c^3y$, &c. denote a Series of Lines, of which the Angles of the ſubſequents, made with their Abſciſs's at the Vertices, are always infinitely leſs than the Angles of the preceding. Moreover, between the Angles of Contact of any two of theſe kinds, other Angles of Contact may be found *ad infinitum*, that ſhall infinitely exceed each other.

41. Now it appears, that Angles of Contact of one kind are infinitely greater than thoſe of another kind; ſince a Curve of one kind, however great it may be, cannot, at the Point of Contact,

lie between the Tangent and a Curve of another kind, however ſmall that Curve may be. Or an Angle of Contact of one kind cannot neceſſarily contain an Angle of Contact of another kind, as the whole contains a part. Thus the Angle of Contact of the Curve $x^4 = cy^3$, or the Angle which it makes with its Abſciſs, neceſſarily includes the Angle of Contact of the Curve $x^3 = by^2$, and can never be contain'd by it. For Angles that can mutually exceed each other are of the ſame kind, as it happens with the aforeſaid Angles of the Trochoid, and of this Curve $x^3 = by^2$.

42. And hence it appears, that Curves, in ſome Points, may be infinitely more ſtraight, or infinitely more curved, than any Circle, and yet not, on that account, loſe the form of Curve-lines. But all this by the way only.



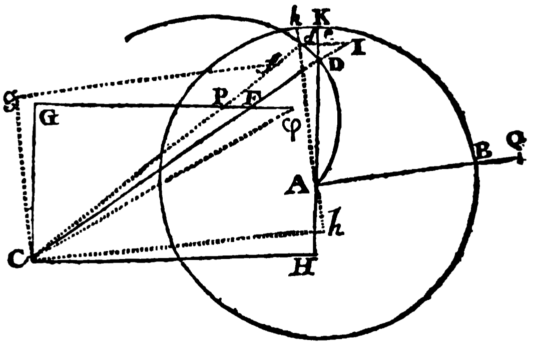
43. Ex. 5. Let ED be the Quadratrix to the Circle, deſcribed from Center A; and letting fall DB perpendicular to AE, make AB $= x$, BD $= y$, and AE $= 1$. Then 'twill be $\dot{y}x - \dot{y}y^2 - \dot{y}x^2 = \dot{x}y$, as before. Then writing 1 for $\dot{x}$, and z for $\dot{y}$, the Equation becomes $zx - zy^2 - zx^2 = y$; and thence, by Prob. 1. $\dot{z}x - \dot{z}y^2 - \dot{z}x^2 + z\dot{x} - 2zx\dot{x} - 2zy\dot{y} = \dot{y}$. Then reducing, and again writing 1 for $\dot{x}$, and $\dot{z}$ for $\dot{y}$, there ariſes $\dot{z} = \frac{2z^2y + 2zx}{x - xx - yy}$. But z and $\dot{z}$ being found, make $\frac{1 + zz}{\dot{z}} =$ DH, and draw HC as above.

44. If you deſire a Conſtruction of the Problem, you will find it very ſhort. Thus draw DP perpendicular to DT, meeting AT in P, and make 2AP : AE :: PT : CH. For $z = \left(\frac{y}{x - xx - yy} =\right) \frac{BD}{-BT}$, and $zy = \frac{BDq}{-BT} = -BP$; and $zy + x = -AP$, and $\frac{2z}{x - xx - yy}$ into $zy + x = \frac{2BD}{AE \times BTq}$ into $-AP = \dot{z}$. Moreover it is $1 + zz = \frac{PT}{BT}$, (becauſe $= 1 + \frac{BDq}{BTq} = \frac{DTq}{BTq}$,) and therefore $\frac{1 + zz}{\dot{z}} = \frac{PT \times AE \times BT}{-2BD \times AP} =$ DH. Laſtly, it is BT : BD :: DH : CH $= \frac{PT \times AE}{-2AP}$. Here the negative Value only ſhews, that CH muſt be taken the ſame way as AB from DH.

45. In the ſame manner the Curvature of Spirals, or of any other Curves whatever, may be determined by a very ſhort Calculation.

46. Furthermore, to determine the Curvature without any previous reduction, when the Curves are refer'd to right Lines in any other manner, this Method might have been apply'd, as has been done already for drawing Tangents. But as all Geometrical Curves, as alſo Mechanical, (eſpecially when the defining conditions are reduced to infinite Equations, as I ſhall ſhew hereafter,) may be refer'd to rectangular Ordinates, I think I have done enough in this matter. He that deſires more, may eaſily ſupply it by his own induſtry; eſpecially if for a farther illuſtration I ſhall add the Method for Spirals.

47. Let BK be a Circle, A its Center, and B a given Point in its Circumference. Let AD*d* be a Spiral, DC its Perpendicular, and C the Center of Curvature at the Point D. Then drawing the right Line ADK, and CG parallel and equal to AK, as alſo the Perpendicular GF meeting CD in F: Make AB or AK $=$ 1 $=$ CG, BK $= x$, AD $= y$, and GF $= z$. Then conceive the Point D to move in the Spiral for an infinitely little Space D*d*, and then through *d* draw the Semidiameter A*k*, and C*g* parallel and equal to it, draw *gf* perpendicular to *g*C, ſo that C*d* cuts *gf* in *f*, and GF in P; produce GF to φ, ſo that Gφ $=$ *gf*, and draw *de* perpendicular to AK, and produce it till it meets CD at I. Then the contemporaneous Moments of BK, AD, and Gφ, will be K*k*, D*e* and Fφ, which therefore may be call'd $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$.



48. Now it is AK : A*e* (AD) :: *k*K : $de = yo$, where I aſſume $\dot{x} = 1$, as above. Alſo CG : GF :: *de* : $eD = oyz$, and therefore $yz = \dot{y}$. Beſides CG : CF :: *de* : $dD = oy \times CF$ :: *d*D : $dI = oy \times CFq$. Moreover, becauſe ∠PCφ (= ∠GC*g*) = ∠DA*d*, and ∠CPφ (= ∠C*d*I = ∠ *ed*D + Rect.) = ∠ AD*d*, the Triangles CPφ and AD*d* are ſimilar, and thence AD : D*d* :: CP (CF) : $P\varphi = o \times CFq$. From whence take Fφ, and there will remain PF $= o \times CFq - o \times \dot{z}$. Laſtly, letting fall CH perpendicular to AD, 'tis PF : *d*I :: CG : *e*H or $DH = \frac{y \times CFq}{CFq - \dot{z}}$. Or ſubſtituting $1 + zz$ for CF*q*, 'twill be $DH = \frac{y + yzz}{1 + zz - \dot{z}}$. Here it may be obſerved,

that in this kind of Computations, I take thoſe Quantities (AD and A*e*) for equal, the Ratio of which differs but infinitely little from the Ratio of Equality.

49. Now from hence ariſes the following Rule. The Relation of x and y being exhibited by any Equation, find the Relation of the Fluxions $\dot{x}$ and $\dot{y}$, (by Prob. 1.) and ſubſtitute 1 for $\dot{x}$, and yz for $\dot{y}$. Then from the reſulting Equation find again, (by Prob. 1.) the Relation between $\dot{x}$, $\dot{y}$, and $\dot{z}$, and again ſubſtitute 1 for $\dot{x}$. The firſt reſult by due reduction will give y and z, and the latter will give $\dot{z}$; which being known, make $\frac{y+yzz}{1+zz-\dot{z}} = \text{DH}$, and raiſe the Perpendicular HC, meeting the Perpendicular to the Spiral DC before drawn in C, and C will be the Center of Curvature. Or which comes to the ſame thing, take CH : HD :: z : 1, and draw CD.

50. Ex. 1. If the Equation be $ax = y$, (which will belong to the Spiral of *Archimedes*,) then (by Prob. 1.) $a\dot{x} = \dot{y}$, or (writing 1 for $\dot{x}$, and yz for $\dot{y}$,) $a = yz$. And hence again (by Prob 1.) $0 = \dot{y}z + y\dot{z}$. Wherefore any Point D of the Spiral being given, and thence the length AD or y, there will be given $z = \frac{a}{y}$, and $\dot{z} = (-\frac{\dot{y}z}{y}$ or$) - \frac{az}{y}$. Which being known, make $1 + zz - \dot{z}$: $1 + zz$ :: DA (y) : DH. And 1 : z :: DH : CH.

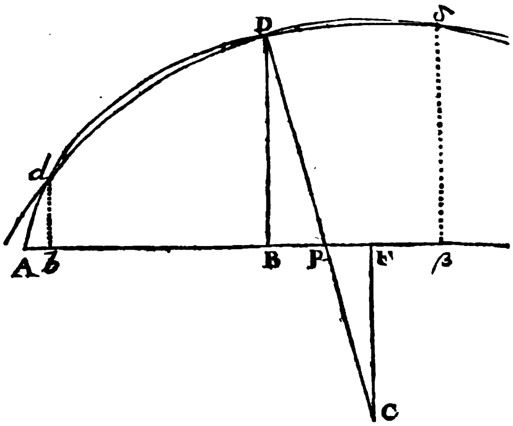
And hence you will eaſily deduce the following Conſtruction. Produce AB to Q, ſo that AB : Arch BK :: Arch BK : BQ, and make AB + AQ : AQ :: DA : DH :: a : HC.

51. Ex. 2. If $ax^2 = y^3$ be the Equation that determines the Relation between BK and AD; (by Prob. 1.) you will have $2ax\dot{x} = 3\dot{y}y^2$, or $2ax = 3zy^3$. Thence again $2a\dot{x} = 3\dot{z}y^3 + 9z\dot{y}y^2$. 'Tis therefore $z = \frac{2ax}{3y^3}$, and $\dot{z} = \frac{2a - 9zzy^3}{3y^3}$. Theſe being known, make $1 + zz - \dot{z}$: $1 + zz$ :: DA : DH. Or, the work being reduced to a better form, make $9xx + 10$: $9xx + 4$ :: DA : DH.

52. Ex. 3. After the ſame manner, if $ax^2 - bxy = y^3$ determines the Relation of BK to AD; there will ariſe $\frac{2ax - by}{bxy + 3y^3} = z$, and $\frac{2a - 2bzy - bz^2xy - 9z^2y^3}{bxy + 3y^3} = \dot{z}$. From which DH, and thence the Point C, is determined as before.

53. And thus you will eaſily determine the Curvature of any other Spirals; or invent Rules for any other kinds of Curves, in imitation of theſe already given.

54. And now I have finiſh'd the Problem; but having made uſe of a Method which is pretty different from the common ways of operation, and as the Problem itſelf is of the number of thoſe which are not very frequent among Geometricians: For the illuſtration and confirmation of the Solution here given, I ſhall not think much to give a hint of another, which is more obvious, and has a nearer relation to the uſual Methods of drawing Tangents. Thus if from any Center, and with any Radius, a Circle be conceived to be deſcribed, which may cut any Curve in ſeveral Points; if that Circle be ſuppos'd to be contracted, or enlarged, till two of the Points of interſection coincide, it will there touch the Curve. And beſides, if its Center be ſuppos'd to approach towards, or recede from, the Point of Contact, till the third Point of interſection ſhall meet with the former in the Point of Contact; then will that Circle be æquicurved with the Curve in that Point of Contact: In like manner as I inſinuated before, in the laſt of the five Properties of the Center of Curvature, by the help of each of which I affirm'd the Problem might be ſolved in a different manner.



55. Therefore with Center C, and Radius CD, let a Circle be deſcribed, that cuts the Curve in the Points d, D, and δ; and letting fall the Perpendiculars DB, db, $\delta\beta$, and CF, to the Abſciſs AB; call AB $= x$, BD $= y$, AF $= v$, FC $= t$, and DC $= s$. Then BF $= v - x$, and DB $+$ FC $= y + t$. The ſum of the Squares of theſe is equal to the Square of DC; that is, $v^2 - 2vx + x^2 + y^2 + 2yt + t^2 = ss$. If you would abbreviate this, make $v^2 + t^2 - s^2 = q^2$, (any Symbol at pleaſure,) and it becomes $x^2 - 2vx + y^2 + 2ty + q^2 = 0$. After you have found t, v, and q^2, you will have $s = \sqrt{v^2 + t^2 - q^2}$.

56. Now let any Equation be propoſed for defining the Curve, the quantity of whoſe Curvature is to be found. By the help of this Equation you may exterminate either of the Quantities x or y,

and there will ariſe an Equation, the Roots of which, (db, DB, $\delta\beta$, &c. if you exterminate x; or Ab, AB, Aβ, &c. if you exterminate y,) are at the Points of interſection d, D, δ, &c. Wherefore ſince three of them become equal, the Circle both touches the Curve, and will alſo be of the ſame degree of Curvature as the Curve, in the point of Contact. But they will become equal by comparing the Equation with another fictitious Equation of the ſame number of Dimenſions, which has three equal Roots; as *Des Cartes* has ſhew'd. Or more expeditiouſly by multiplying its Terms twice by an Arithmetical Progreſſion.

57. EXAMPLE. Let the Equation be $ax = yy$, (which is an Equation to the Parabola,) and exterminating x, (that is, ſubſtituting its Value $\frac{yy}{a}$ in the foregoing Equation,) there will ariſe

$$\begin{array}{cccccc} \frac{y^4}{aa} & * & -\frac{2v}{a}y^2 & +\ 2ty & +\ q^2 & = 0. \\ & & +\ y^2 & & & \\ 4 & * & 2 & 1 & 0 & \\ 3 & * & 1 & 0 & -1 & \\ \hline \frac{12y^4}{aa} & & -\frac{4v}{a}y^2 & +\ 2y^2 & = 0. & \end{array}$$

Three of whoſe Roots y are to be made equal. And for this purpoſe I multiply the Terms twice by an Arithmetical Progreſſion, as you ſee done here; and there ariſes $\frac{12y^4}{aa} - \frac{4v}{a}y^2 + 2y^2 = 0$. Or $v = \frac{3y^2}{a} + \frac{1}{2}a$. Whence it is eaſily infer'd, that BF $= 2x + \frac{1}{2}a$, as before.

58. Wherefore any Point D of the Parabola being given, draw the Perpendicular DP to the Curve, and in the Axis take PF $= 2$AB, and erect FC Perpendicular to FA, meeting DP in C; then will C be the Center of Curvity deſired.

59. The ſame may be perform'd in the Ellipſis and Hyperbola, but the Calculation will be troubleſome enough, and in other Curves generally very tedious.

Of Queſtions that have ſome Affinity to the preceding Problem.

60. From the Reſolution of the preceding Problem ſome others may be perform'd; ſuch are,

I. *To find the Point where the Curve has a given degree of Curvature.*

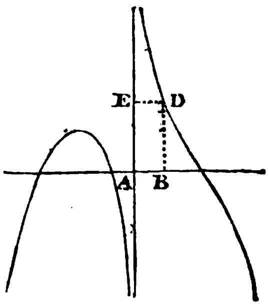
61. Thus in the Parabola, $ax = yy$, if the Point be required whoſe Radius of Curvature is of a given length f: From the Center of Curvature, found as before, you will determine the Radius

to be $\frac{a+4x}{2a}\sqrt{aa+4ax}$, which muſt be made equal to f. Then by reduction there ariſes $x = -\frac{1}{4}a + \sqrt[3]{\frac{1}{16}aff}$.

II. *To find the Point of Rectitude.*

62. I call that the *Point of Rectitude*, in which the Radius of Flexure becomes infinite, or its Center at an infinite diſtance: Such it is at the Vertex of the Parabola $a^3x = y^4$. And this ſame Point is commonly the Limit of contrary Flexure, whoſe Determination I have exhibited before. But another Determination, and that not inelegant, may be derived from this Problem. Which is, the longer the Radius of Flexure is, ſo much the leſs the Angle DC*d* (*Fig. pag.* 61.) becomes, and alſo the Moment δf; ſo that the Fluxion of the Quantity z is diminiſh'd along with it, and by the Infinitude of that Radius, altogether vaniſhes. Therefore find the Fluxion $\dot{z}$, and ſuppoſe it to become nothing.

63. As if we would determine the Limit of contrary Flexure in the Parabola of the ſecond kind, by the help of which *Carteſius* conſtructed Equations of ſix Dimenſions; the Equation to that Curve is $x^3 - bx^2 - cdx + bcd + dxy = 0$. And hence (by Prob. 1.) ariſes $3\dot{x}x^2 - 2b\dot{x}x - cd\dot{x} + d\dot{x}y + dx\dot{y} = 0$. Now writing 1 for $\dot{x}$, and z for $\dot{y}$, it becomes $3x^2 - 2bx - cd + dy + dxz = 0$; whence again (by Prob. 1.) $6\dot{x}x - 2b\dot{x} + d\dot{y} + d\dot{x}z + dx\dot{z} = 0$. Here again writing 1 for $\dot{x}$, z for $\dot{y}$, and 0 for $\dot{z}$, it becomes $6x - 2b + 2dz = 0$. And exterminating z, by putting $b - 3x$ for dz in the Equation $3xx - 2bx - cd + dy + dxz = 0$, there will ariſe $-bx - cd + dy = 0$, or $y = c + \frac{bx}{d}$; this being ſubſtituted in the room of y in the Equation of the Curve, we ſhall have $x^3 + bcd = 0$; which will determine the Confine of contrary Flexure.



64. By a like Method you may determine the Points of Rectitude, which do not come between parts of contrary Flexure. As if the Equation $x^4 - 4ax^3 + 6a^2x^2 - b^3y = 0$ expreſs'd the nature of a Curve; you have firſt, (by Prob. 1.) $4x^3 - 12ax^2 + 12a^2x - b^3z = 0$, and hence again $12x^2 - 24ax + 12a^2 - b^3\dot{z} = 0$. Here ſuppoſe $\dot{z} = 0$, and by Reduction there will ariſe $x = a$. Wherefore take $AB = a$, and erect the perpendicular BD; this will meet the Curve in the Point of Rectitude D, as was required.

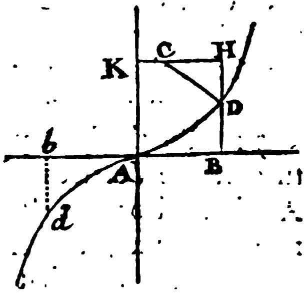
III.

III. *To find the Point of infinite Flexure.*

65. Find the Radius of Curvature, and ſuppoſe it to be nothing. Thus to the Parabola of the ſecond kind, whoſe Equation is $x^3 = ay^2$, that Radius will be $CD = \frac{4a+9x}{6a}\sqrt{4ax+9xx}$; which becomes nothing when $x = 0$.

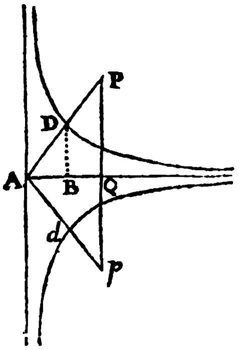
IV. *To determine the Point of the greateſt or leaſt Flexure.*

66. At theſe Points the Radius of Curvature becomes either the greateſt or leaſt. Wherefore the Center of Curvature, at that moment of Time, neither moves towards the point of Contact, nor the contrary way, but is intirely at reſt. Therefore let the Fluxion of the Radius CD be found; or more expeditiouſly, let the Fluxion of either of the Lines BH or AK be found, and let it be made equal to nothing.



67. As if the Queſtion were propoſed concerning the Parabola of the ſecond kind $x^3 = a^2y$; firſt to determine the Center of Curvature you will find $DH = \frac{aa+9xy}{6x}$, and therefore $BH = \frac{aa+15xy}{6x}$; make $BH = v$, then $\frac{aa}{6x} + \frac{5}{2}y = v$. Hence (by Prob. 1.) $-\frac{a^2\dot{x}}{6xx} + \frac{5}{2}\dot{y} = \dot{v}$. But now ſuppoſe $\dot{v}$, or the Fluxion of BH, to be nothing; and beſides, ſince by Hypotheſis $x^3 = a^2y$, and thence (by Prob. 1.) $3\dot{x}x^2 = a^2\dot{y}$, putting $\dot{x} = 1$, ſubſtitute $\frac{3xx}{aa}$ for $\dot{y}$, and there will ariſe $45x^4 = a^4$. Take therefore $AB = a\sqrt[4]{\frac{1}{45}} = a \times \overline{45}|^{-\frac{1}{4}}$, and raiſing the perpendicular BD, it will meet the Curve in the Point of the greateſt Curvature. Or, which is the ſame thing, make $AB : BD :: 3\sqrt{5} : 1$.

68. After the ſame manner the Hyperbola of the ſecond kind repreſented by the Equation $xy^2 = a^3$, will be moſt inflected in the points D and *d*, which you may determine by taking in the Abſciſs $AQ = 1$, and erecting the Perpendicular $QP = \sqrt{5}$, and Qp equal to it on the other ſide. Then drawing AP and A*p*, they will meet the Curve in the points D and *d* required.



V. *To determine the Locus of the Center of Curvature, or to deſcribe the Curve, in which that Center is always found.*

69. We have already ſhewn, that the Center of Curvature of the Trochoid is always found in another Trochoid. And thus the Center of Curvature of the Parabola is found in another Parabola of the ſecond kind, repreſented by the Equation $axx = y^3$, as will eaſily appear from Calculation.

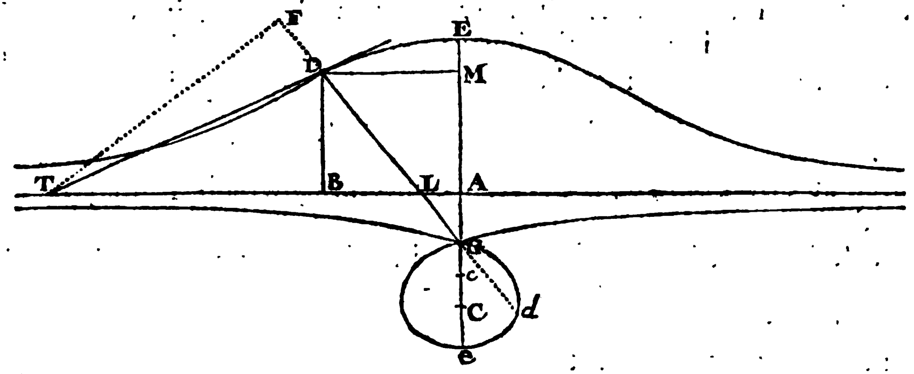
VI. *Light falling upon any Curve, to find its Focus, or the Concourſe of the Rays that are refracted at any of its Points.*

70. Find the Curvature at that Point of the Curve, and deſcribe a Circle from the Center, and with the Radius of Curvature. Then find the Concourſe of the Rays, when they are refracted by a Circle about that Point: For the ſame is the Concourſe of the refracted Rays in the propoſed Curve.

71. To theſe may be added a particular Invention of the Curvature at the Vertices of Curves, where they cut their Abſciſſes at right Angles. For the Point in which the Perpendicular to the Curve, meeting with the Abſciſs, cuts it ultimately, is the Center of its Curvature. So that having the relation between the Abſciſs x, and the rectangular Ordinate y, and thence (by Prob. 1.) the relation between the Fluxions $\dot{x}$ and $\dot{y}$; the Value $y\dot{y}$, if you ſubſtitute 1 for $\dot{x}$ into it, and make $y = 0$, will be the Radius of Curvature.

72. Thus in the Ellipſis $ax - \frac{a}{b}xx = yy$, it is $\frac{a\dot{x}}{2} - \frac{ax\dot{x}}{b} = y\dot{y}$; which Value of $y\dot{y}$, if we ſuppoſe $y = 0$, and conſequently $x = b$, writing 1 for $\dot{x}$, becomes $\frac{1}{2}a$ for the Radius of Curvature. And ſo at the Vertices of the Hyperbola and Parabola, the Radius of Curvature will be always half of the Latus rectum.

73. And in like manner for the Conchoid, defined by the Equation $\frac{b^2c^2}{xx} + \frac{2bcc}{x} \begin{smallmatrix}+cc\\-bb\end{smallmatrix} - 2bx - xx = yy$, the Value of yy, (found by

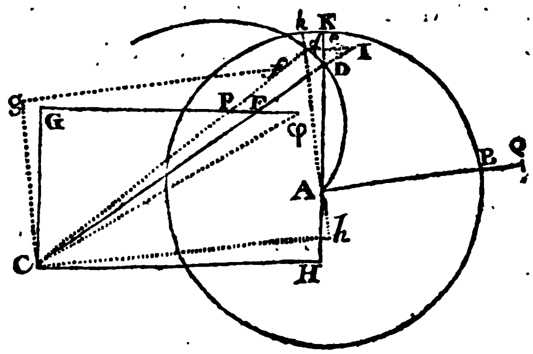


Prob. I.) will be $-\frac{b^2c^2}{x^3} - \frac{bc^2}{x^2} - b - x$. Now ſuppoſing $y = 0$, and thence $x = c$ or $-c$, we ſhall have $-\frac{bb}{c} - 2b - c$, or $\frac{bb}{c} - 2b + c$, for the Radius of Curvature. Therefore make AE : EG :: EG : EC, and A*e* : *e*G :: *e*G : *ec*, and you will have the Centers of Curvature C and *c*, at the Vertices E and *e* of the Conjugate Conchoids.

PROB. VI.

To determine the Quality of the Curvature, at a given Point of any Curve.

1. By the *Quality of Curvature* I mean its Form, as it is more or leſs inequable, or as it is varied more or leſs, in its progreſs thro' different parts of the Curve. So if it were demanded, what is the Quality of the Curvature of the Circle? it might be anſwer'd, that it is uniform, or invariable. And thus if it were demanded, what is the Quality of the Curvature of the Spiral, which is deſcribed by the motion of the point D, proceeding from A in AD with an accelerated velocity, while the right Line AK moves with an uniform rotation about the Center A; the acceleration of



which Velocity is ſuch, that the right Line AD has the ſame ratio to the Arch BK, deſcribed from a given point B, as a Number has to its Logarithm: I ſay, if it be aſk'd, What is the Quality of the Curvature of this Spiral? It may be anſwer'd, that it is uniformly varied, or that it is equably inequable. And thus other Curves, in their ſeveral Points, may be denominated inequably inequable, according to the variation of their Curvature.

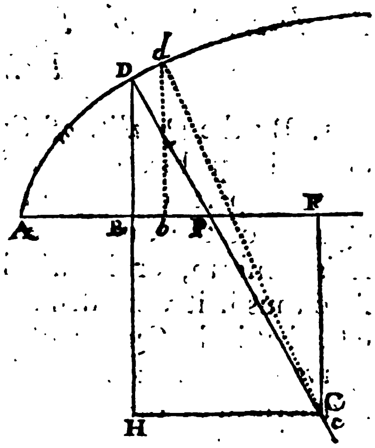
2. Therefore the Inequability or Variation of Curvature is required at any Point of a Curve. Concerning which it may be obſerved,

3. I. That at Points placed alike in like Curves, there is a like Inequability or Variation of Curvature.

4. II. And that the Moments of the Radii of Curvature, at thoſe Points, are proportional to the contemporaneous Moments of the Curves, and the Fluxions to the Fluxions.

5. III. And therefore, that where thoſe Fluxions are not proportional, the Inequability of the Curvature will be unlike. For there will be a greater Inequability, where the Ratio of the Fluxion of the Radius of Curvature to the Fluxion of the Curve is greater. And therefore that ratio of the Fluxions may not improperly be call'd the Index of the Inequability or of the Variation of Curvature.

6. At the points D and *d*, infinitely near to each other, in the Curve AD*d*, let there be drawn the Radii of Curvature DC and *dc*; and D*d* being the Moment of the Curve, C*c* will be the contemporaneous Moment of the Radius of Curvature, and $\frac{Cc}{Dd}$ will be the Index of the Inequability of Curvature. For the Inequability may be call'd ſuch and ſo great, as the quantity of that ratio $\frac{Cc}{Dd}$ ſhews it to be: Or the Curvature may be ſaid to be ſo much the more unlike to the uniform Curvature of a Circle.



7. Now letting fall the perpendicular Ordinates DB and *db*, to any line AB meeting DC in P; make AB $= x$, BD $= y$, DP $= t$, DC $= v$, and thence B*b* $= \dot{x}o$, it will be C*c* $= \dot{v}o$; and BD : DP :: B*b* : D*d* $= \frac{\dot{x}ot}{y}$, and $\frac{Cc}{Dd} = \frac{\dot{v}}{\frac{\dot{x}t}{y}} = \frac{\dot{v}y}{t}$, making $\dot{x} = 1$.

Wherefore the relation between x and y being exhibited by any Equation, and thence, (according to Prob. 4. and 5.) the Perpendicular DP or t, being found, and the Radius of Curvature v, and the Fluxion $\dot{v}$ of that Radius, (by Prob. 1.) the Index $\frac{\dot{v}y}{t}$ of the Inequability of Curvature will be given alſo.

8. Ex. 1. Let the Equation to the Parabola $2ax=yy$ be given; then (by Prob. 4.) BP $=a$, and therefore DP $=\sqrt{aa+yy}=t$. Alſo (by Prob. 5.) BF $=a+2x$, and BP : DP :: BF : DC $=\frac{at+2tx}{a}=v$. Now the Equations $2ax=yy$, $aa+yy=tt$, and $\frac{at+2tx}{a}=v$, (by Prob. 1.) give $2a\dot{x}=2\dot{y}y$, and $2\dot{y}y=2\dot{t}t$, and $\frac{a\dot{t}+2\dot{t}x+2t\dot{x}}{a}=\dot{v}$. Which being reduced to order, and putting $\dot{x}=1$, there will ariſe $\dot{y}=\frac{a}{y}$, $\dot{t}=\left(\frac{\dot{y}y}{t}=\right)\frac{a}{t}$, and $\dot{v}=\frac{a\dot{t}+2\dot{t}x+2t}{a}$. And thus $\dot{y}$, $\dot{t}$, and $\dot{v}$ being found, there will be had $\frac{\dot{v}y}{t}$ the Index of the Inequability of Curvature.

9. As if in Numbers it were determin'd, that $a=1$, or $2x=yy$, and $x=\frac{1}{2}$; then $y\left(=\sqrt{2x}\right)=1$, $\dot{y}\left(=\frac{a}{y}\right)=1$, $t\left(=\sqrt{aa+yy}=\right)\sqrt{2}$, $\dot{t}\left(=\frac{a}{t}\right)=\sqrt{\frac{1}{2}}$, and $\dot{v}=\left(\frac{a\dot{t}+2\dot{t}x+2t}{a}=\right)3\sqrt{2}$. So that $\frac{\dot{v}y}{t}=3$, which therefore is the Index of Inequability.

10. But if it were determin'd, that $x=2$, then $y=2$, $\dot{y}=\frac{1}{2}$, $t=\sqrt{5}$, $\dot{t}=\sqrt{\frac{1}{5}}$, and $\dot{v}=3\sqrt{5}$. So that $\left(\frac{\dot{v}y}{t}=\right)6$ will be here the Index of Inequability.

11. Wherefore the Inequability of Curvature at the Point of the Curve, from whence an Ordinate, equal to the Latus rectum of the Parabola, being drawn perpendicular to the Axis, will be double to the Inequability at that Point, from whence the Ordinate ſo drawn is half the Latus rectum; that is, the Curvature at the firſt Point is as unlike again to the Curvature of the Circle, as the Curvature at the ſecond Point.

12. Ex. 2. Let the Equation be $2ax-bxx=yy$, and (by Prob. 4.) it will be $a-bx=$ BP, and thence $tt=(aa-2abx+bbxx+yy=)\ aa-byy+yy$. Alſo (by Prob. 5.) it is DH $=y+\frac{y^3-by^3}{aa}$; where, if for $yy-byy$ you ſubſtitute $tt-aa$, there ariſes DH $=\frac{tty}{aa}$. 'Tis alſo BD : DP :: DH : DC $=\frac{t^3}{a^2}=v$. Now (by Prob. 1.) the Equations $2ax-bxx=yy$, $aa-byy+yy=tt$, and $\frac{t^3}{aa}=v$, give

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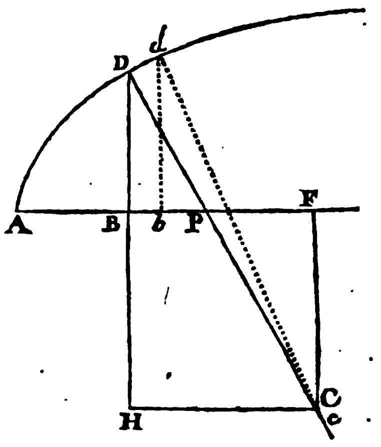
give $a - bx = \dot{y}y$, and $\dot{y}y - b\dot{y}y = tt$, and $\frac{3\dot{t}tt}{aa} = \dot{v}$. And thus $\dot{v}$ being found, the Index $\frac{\dot{v}y}{t}$ of the Inequability of Curvature, will alſo be known.



13. Thus in the Ellipſis $2x - 3xx = yy$, where it is $a = 1$, and $b = 3$; if we make $x = \frac{1}{2}$, then $y = \frac{1}{2}$, $\dot{y} = -1$, $t = \sqrt{\frac{1}{2}}$, $\dot{t} = \sqrt{2}$, $\dot{v} = 3\sqrt{\frac{1}{2}}$, and therefore $\frac{\dot{v}y}{t} = \frac{3}{2}$, which is the Index of the Inequability of Curvature. Hence it appears, that the Curvature of this Ellipſis, at the Point D here aſſign'd, is by two times leſs inequable, (or by two times more like to the Curvature of the Circle,) than the Curvature of the Parabola, at that Point of its Curve, from whence an Ordinate let fall upon the Axis is equal to half the Latus rectum.

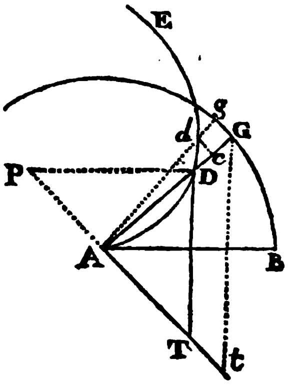
14. If we have a mind to compare the Concluſions derived in theſe Examples, in the Parabola $2ax = yy$ ariſes $(\frac{\dot{v}y}{t} =) \frac{3y}{a}$ for the Index of Inequability; and in the Ellipſis $2ax - bxx = yy$, ariſes $(\frac{\dot{v}y}{t} =) \frac{3y - 3by}{aa} \times BP$; and ſo in the Hyperbola $2ax + bxx = yy$, the analogy being obſerved, there ariſes the Index $(\frac{\dot{v}y}{t} =) \frac{3y + 3by}{aa} \times BP$. Whence it is evident, that at the different Points of any Conic Section conſider'd apart, the Inequability of Curvature is as the Rectangle BD × BP. And that, at the ſeveral Points of the Parabola, it is as the Ordinate BD.

15. Now as the Parabola is the moſt ſimple Figure of thoſe that are curved with inequable Curvature, and as the Inequability of its Curvature is ſo eaſily determined, (for its Index is $6 \times \frac{\text{Ordinate}}{\text{Lat. rect.}}$,) therefore the Curvatures of other Curves may not improperly be compared to the Curvature of this.

16. As if it were inquired, what may be the Curvature of the Ellipſis $2x - 3xx = yy$, at that Point of the Perimeter which is determined by aſſuming $x = \frac{1}{2}$: Becauſe its Index is $\frac{3}{2}$, as before, it might be anſwer'd, that it is like the Curvature of the Parabola

$6x = yy$, at that Point of the Curve, between which and the Axis the perpendicular Ordinate is equal to $\frac{3}{2}$.

17. Thus, as the Fluxion of the Spiral ADE is to the Fluxion of the Subtenſe AD, in a certain given Ratio, ſuppoſe as d to e; on its concave ſide erect $AP = \frac{e}{\sqrt{dd-ee}} \times AD$ perpendicular to AD, and P will be the Center of Curvature, and $\frac{AP}{AD}$ or $\frac{e}{\sqrt{dd-ee}}$, will be the Index of Inequability. So that this Spiral has every where its Curvature alike inequable, as the Parabola $6x = yy$ has in that Point of its Curve, from whence to its Abſciſs a perpendicular Ordinate is let fall, which is equal to the quantity $\frac{e}{\sqrt{dd-ee}}$.



18. And thus the Index of Inequability at any Point D of the Trochoid, (ſee Fig. in *Art.* 29. *pag.* 64.) is found to be $\frac{AB}{BL}$. Wherefore its Curvature at the ſame Point D is as inequable, or as unlike to that of a Circle, as the Curvature of any Parabola $ax = yy$ is at the Point where the Ordinate is $\frac{1}{6}a \times \frac{AB}{BL}$.

19. And from theſe Conſiderations the Senſe of the Problem, as I conceive, muſt be plain enough; which being well underſtood, it will not be difficult for any one, who obſerves the Series of the things above deliver'd, to furniſh himſelf with more Examples, and to contrive many other Methods of operation, as occaſion may require. So that he will be able to manage Problems of a like nature, (where he is not diſcouraged by tedious and perplex Calculations,) with little or no difficulty. Such are theſe following;

I. *To find the Point of any Curve, where there is either no Inequability of Curvature, or infinite, or the greateſt, or the leaſt.*

20. Thus at the Vertices of the Conic Sections, there is no Inequability of Curvature; at the Cuſpid of the Trochoid it is infinite; and it is greateſt at thoſe Points of the Ellipſis, where the Rectangle BD × BP is greateſt, that is, where the Diagonal-Lines of the circumſcribed Parallelogram cut the Ellipſis, whoſe Sides touch it in their principal Vertices.

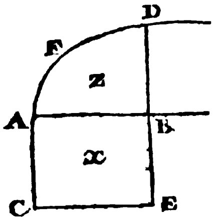
II. *To determine a Curve of ſome definite Species, ſuppoſe a Conic Section, whoſe Curvature at any Point may be equal and ſimilar to the Curvature of any other Curve, at a given Point of it.*

III. *To determine a Conic Section, at any Point of which, the Curvature and Poſition of the Tangent, (in reſpect of the Axis,) may be like to the Curvature and Poſition of the Tangent, at a Point aſſign'd of any other Curve.*

21. The uſe of which Problem is this, that inſtead of Ellipſes of the ſecond kind, whoſe Properties of refracting Light are explain'd by *Des Cartes* in his Geometry, Conic Sections may be ſubſtituted, which ſhall perform the ſame thing, very nearly, as to their Refractions. And the ſame may be underſtood of other Curves.

PROB. VII.

To find as many Curves as you pleaſe, whoſe Areas may be exhibited by finite Equations.



1. Let AB be the Abſciſs of a Curve, at whoſe Vertex A let the perpendicular AC $=$ 1 be raiſed, and let CE be drawn parallel to AB. Let alſo DB be a rectangular Ordinate, meeting the right Line CE in E, and the Curve AD in D. And conceive theſe Areas ACEB and ADB to be generated by the right Lines BE and BD, as they move along the Line AB. Then their Increments or Fluxions will be always as the deſcribing Lines BE and BD. Wherefore make the Parallelogram ACEB, or AB × 1, $= x$, and the Area of the Curve ADB call z. And the Fluxions $\dot{x}$ and $\dot{z}$ will be as BE and BD; ſo that making $\dot{x} = 1 =$ BE, then $\dot{z} =$ BD.

2. Now if any Equation be aſſumed at pleaſure, for determining the relation of z and x, from thence, (by Prob. 1.) may $\dot{z}$ be derived. And thus there will be two Equations, the latter of which will determine the Curve, and the former its Area.

EXAMPLES.

3. Aſſume $xx = z$, and thence (by Prob. 1.) $2\dot{x}x = \dot{z}$, or $2x = \dot{z}$, becauſe $\dot{x} = 1$.

4. Aſſume $\frac{x^3}{a} = z$, and thence will ariſe $\frac{3x^2}{a} = \dot{z}$, an Equation to the Parabola.

5. Aſſume $ax^3 = zz$, or $a^{\frac{1}{2}}x^{\frac{3}{2}} = z$, and there will ariſe $\frac{3}{2}a^{\frac{1}{2}}x^{\frac{1}{2}} = \dot{z}$, or $\frac{9}{4}ax = \dot{z}\dot{z}$, an Equation again to the Parabola.

6. Aſſume $a^6x^{-2} = zz$, or $a^3x^{-1} = z$, and there ariſes $-a^3x^{-2} = \dot{z}$, or $a^3 + \dot{z}xx = 0$. Here the negative Value of $\dot{z}$ only inſinuates, that BD is to be taken the contrary way from BE.

7. Again if you aſſume $c^2a^2 + c^2x^2 = z^2$, you will have $2c^2x = 2\dot{z}z$; and z being eliminated, there will ariſe $\frac{cx}{\sqrt{aa+xx}} = \dot{z}$.

8. Or if you aſſume $\frac{aa+xx}{b}\sqrt{aa+xx} = z$, make $\sqrt{aa+xx} = v$, and it will be $\frac{v^3}{b} = z$, and then (by Prob. 1.) $\frac{3\dot{v}vv}{b} = \dot{z}$. Alſo the Equation $aa + xx = vv$ gives $2x = 2v\dot{v}$, by the help of which if you exterminate $\dot{v}$, it will become $\frac{3vx}{b} = \dot{z} = \frac{3x}{b}\sqrt{aa+xx}$.

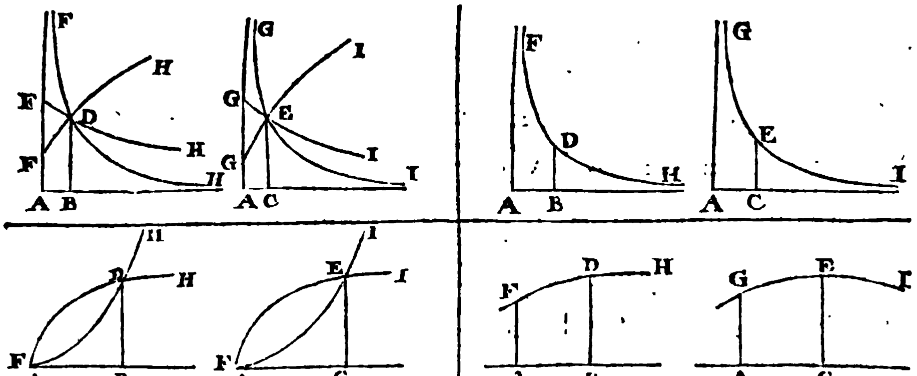
9. Laſtly, if you aſſume $8 - 3xz + \frac{2}{3}z = zz$, you will obtain $-3z - 3x\dot{z} + \frac{2}{3}\dot{z} = 2\dot{z}z$. Wherefore by the aſſumed Equation firſt ſeek the Area z, and then the Ordinate $\dot{z}$ by the reſulting Equation.

10. And thus from the Areas, however they may be feign'd, you may always determine the Ordinates to which they belong.

PROB. VIII.

To find as many Curves as you pleaſe, whoſe Areas ſhall have a relation to the Area of any given Curve, aſſignable by finite Equations.

1. Let FDH be a given Curve, and GEI the Curve required, and conceive their Ordinates DB and EC to move at right Angles upon



their Abſciſſes or Baſes AB and AC. Then the Increments or Fluxions of the Areas which they deſcribe, will be as thoſe Ordinates drawn into their Velocities of moving, that is, into the Fluxions of their Abſciſſes. Therefore make AB $=x$, BD $=v$, AC $=z$, and CE $=y$, the Area AFDB $=s$, and the Area AGEC $=t$, and let the Fluxions of the Areas be $\dot{s}$ and $\dot{t}$: And it will be $\dot{x}v : \dot{z}y :: \dot{s} : \dot{t}$. Therefore if we ſuppoſe $\dot{x}=1$, and $v=\dot{s}$, as before; it will be $\dot{z}y=\dot{t}$, and thence $\frac{\dot{t}}{\dot{z}}=y$.

2. Therefore let any two Equations be aſſumed; one of which may expreſs the relation of the Areas s and t, and the other the relation of their Abſciſſes x and z, and thence, (by Prob. 1.) let the Fluxions $\dot{t}$ and $\dot{z}$ be found, and then make $\frac{\dot{t}}{\dot{z}}=y$.

3. Ex. 1. Let the given Curve FDH be a Circle, expreſs'd by the Equation $ax-xx=vv$, and let other Curves be ſought, whoſe Areas may be equal to that of the Circle. Therefore by the Hypotheſis $s=t$, and thence $\dot{s}=\dot{t}$, and $y=\frac{\dot{t}}{\dot{z}}=\frac{v}{\dot{z}}$. It remains to determine $\dot{z}$, by aſſuming ſome relation between the Abſciſſes x and z.

4. As if you ſuppoſe $ax=zz$; then (by Prob. 1.) $a=2\dot{z}z$: So that ſubſtituting $\frac{a}{2z}$ for $\dot{z}$, then $y=\frac{v}{\dot{z}}=\frac{2vz}{a}$. But it is $v=(\sqrt{ax-xx}=)\ \frac{z}{a}\sqrt{aa-zz}$, therefore $\frac{2zz}{aa}\sqrt{aa-zz}=y$ is the Equation to the Curve, whoſe Area is equal to that of the Circle.

5. After the ſame manner if you ſuppoſe $xx=z$, there will ariſe $2x=\dot{z}$, and thence $y=\left(\frac{v}{\dot{z}}=\right)\frac{v}{2x}$; whence v and x being exterminated, it will be $y=\frac{\sqrt{az^{\frac{1}{2}}-z}}{2z^{\frac{1}{2}}}$.

6. Or if you ſuppoſe $cc=xz$, there ariſes $0=z+x\dot{z}$, and thence $-\frac{vx}{z}=y=-\frac{c^3}{z^3}\sqrt{az-cc}$.

7. Again, ſuppoſing $ax+\frac{s}{1}=z$, (by Prob. 1.) it is $a+\dot{s}=\dot{z}$, and thence $\frac{v}{a+\dot{s}}=y=\frac{v}{a+v}$, which denotes a mechanical Curve.

8. Ex. 2. Let the Circle $ax-xx=vv$ be given again, and let Curves be ſought, whoſe Areas may have any other aſſumed relation to the Area of the Circle. As if you aſſume $cx+s=t$, and ſuppoſe alſo $ax=zz$. (By Prob. 1.) 'tis $c+\dot{s}=\dot{t}$, and $a=2\dot{z}z$.

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Therefore $y = \frac{\dot{t}}{\dot{z}} = \frac{2cz + 2\dot{s}z}{a}$; and ſubſtituting $\sqrt{ax - xx}$ for $\dot{s}$, and $\frac{zz}{a}$ for x, 'tis $y = \frac{2cz}{a} + \frac{4zz}{aa}\sqrt{aa - zz}$.

9. But if you aſſume $s - \frac{2v^3}{3a} = t$, and $x = z$, you will have $\dot{s} - \frac{2\dot{v}v^2}{a} = \dot{t}$, and $1 = \dot{z}$. Therefore $y = \frac{\dot{t}}{\dot{z}} = \dot{s} - \frac{2\dot{v}v^2}{a}$, or $= v - \frac{2\dot{v}v^2}{a}$. Now for exterminating $\dot{v}$, the Equation $ax - xx = vv$, (by Prob. 1.) gives $a - 2\dot{x} = 2\dot{v}v$, and therefore 'tis $y = \frac{2vx}{a}$. Where if you expunge v and x by ſubſtituting their values $\sqrt{ax - xx}$ and z, there will ariſe $y = \frac{2z}{a}\sqrt{az - zz}$.

10. But if you aſſume $ss = t$, and $x = zz$, there will ariſe $2\dot{s}s = \dot{t}$, and $1 = 2\dot{z}z$; and therefore $y = \frac{\dot{t}}{\dot{z}} = 4ssz$. And for $\dot{s}$ and x ſubſtituting $\sqrt{ax - xx}$ and zz, it will become $y = 4szz\sqrt{a - zz}$, which is an Equation to a mechanical Curve.

11. Ex. 3. After the ſame manner Figures may be found, which have an aſſumed relation to any other given Figure. Let the Hyperbola $cc + xx = vv$ be given; then if you aſſume $s = t$, and $xx = cz$, you will have $\dot{s} = \dot{t}$ and $2\dot{x}x = c\dot{z}$; and thence $y = \frac{\dot{t}}{\dot{z}} = \frac{c\dot{s}}{2x}$. Then ſubſtituting $\sqrt{cc + xx}$ for $\dot{s}$, and $c^{\frac{1}{2}}z^{\frac{1}{2}}$ for x, it will be $y = \frac{c}{2z}\sqrt{cz + zz}$.

12. And thus if you aſſume $xv - s = t$, and $xx = cz$, you will have $v + \dot{v}x - \dot{s} = \dot{t}$, and $2x = c\dot{z}$. But $v = \dot{s}$, and thence $\dot{v}x = \dot{t}$. Therefore $y = \frac{\dot{t}}{\dot{z}} = \frac{c\dot{v}}{x}$. But now (by Prob. 1.) $cc + xx = vv$ gives $x = \dot{v}v$, and 'tis $y = \frac{cx}{2v}$. Then ſubſtituting $\sqrt{cc + xx}$ for v, and $c^{\frac{1}{2}}z^{\frac{1}{2}}$ for x, it becomes $y = \frac{cz}{2\sqrt{cz + zz}}$.

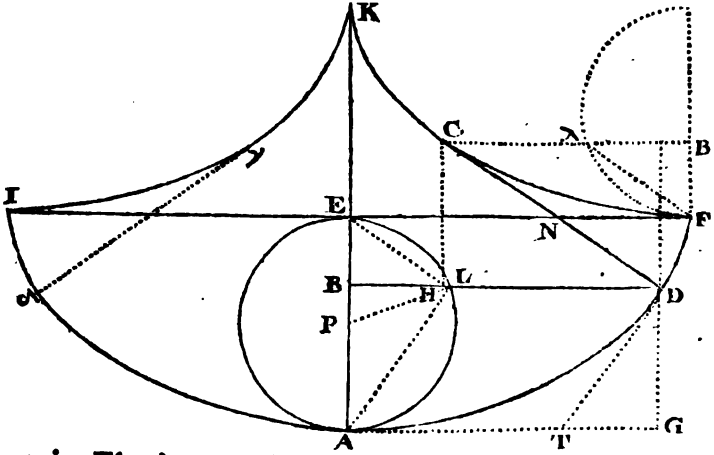
13. Ex. 4. Moreover if the Ciſſoid $\frac{xx}{\sqrt{ax - xx}} = v$ were given, to which other related Figures are to be found, and for that purpoſe you aſſume $\frac{x}{3}\sqrt{ax - xx} + \frac{2}{3}s = t$; ſuppoſe $\frac{x}{3}\sqrt{ax - xx} = b$, and its Fluxion $\dot{b}$; therefore $\dot{b} + \frac{2}{3}\dot{s} = \dot{t}$. But the Equation $\frac{ax^3 - x^4}{9} = bb$

$= bb$ gives $\frac{3ax^2 - 4x^3}{9} = 2bb$, where if you exterminate b, it will be $\dot{b} = \frac{3ax - 4xx}{6\sqrt{ax - xx}}$. And beſides ſince it is $\frac{2}{3}\dot{s} = \frac{2}{3}v = \frac{4xx}{6\sqrt{ax - xx}}$, 'twill be $\frac{ax}{2\sqrt{ax - xx}} = \dot{t}$. Now to determine z and $\dot{z}$, aſſume $\sqrt{aa - ax} = z$; then (by Prob. 1.) $-a = 2\dot{z}z$, or $\dot{z} = -\frac{a}{2z}$. Wherefore it is $y = (\frac{\dot{t}}{\dot{z}} = \frac{-zx}{\sqrt{ax - xx}} = \sqrt{\frac{zzx}{a - x}} = \sqrt{ax} =) \sqrt{aa - zz}$. And as this Equation belongs to the Circle, we ſhall have the relation of the Areas of the Circle and of the Ciſſoid.

14. And thus if you had aſſumed $\frac{2x}{3}\sqrt{ax - xx} + \frac{1}{3}s = t$, and $x = z$, there would have been derived $y = \sqrt{az - zz}$, an Equation again to the Circle.

15. In like manner if any mechanical Curve were given, other mechanical Curves related to it might be found. But to derive geometrical Curves, it will be convenient, that of right Lines depending Geometrically on each other, ſome one may be taken for the Baſe or Abſciſs; and that the Area which compleats the Parallelogram be ſought, by ſuppoſing its Fluxion to be equivalent to the Abſciſs, drawn into the Fluxion of the Ordinate.

16. Ex. 5. Thus the Trochoid ADF being propoſed, I refer it to the Abſciſs AB; and the Parallelogram ABDG being compleated, I ſeek for the complemental Superficies ADG, by ſuppoſing it to be deſcribed by the Motion of the right Line GD, and therefore its Fluxion to be equivalent to the Line GD drawn into the Velocity of the Motion; that is, $x \times v$. Now whereas AL is parallel to the Tangent DT, therefore AB will be to BL as the Fluxion of the ſame AB to the Fluxion of the Ordinate BD, that

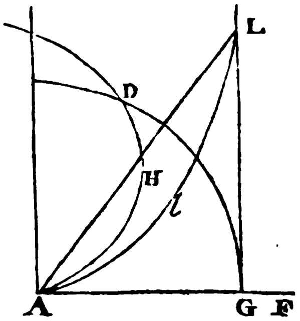


that is, as 1 to $\dot v$. So that $\dot v = \frac{BL}{AB}$, and therefore $x\dot v = BL$. Therefore the Area ADG is deſcribed by the Fluxion BL; ſince therefore the circular Area ALB is deſcribed by the ſame Fluxion, they will be equal.

17. In like manner if you conceive ADF to be a Figure of Arches, or of verſed Sines, that is, whoſe Ordinate BD is equal to the Arch AL; ſince the Fluxion of the Arch AL is to the Fluxion of the Abſciſs AB, as PL to BL, that is, $\dot v : 1 :: \frac{1}{2}a : \sqrt{ax - xx}$, then $\dot v = \frac{a}{2\sqrt{ax - xx}}$. Then $\dot v x$, the Fluxion of the Area ADG, will be $\frac{ax}{2\sqrt{ax - xx}}$. Wherefore if a right Line equal to $\frac{ax}{2\sqrt{ax - xx}}$ be conceived to be apply'd as a rectangular Ordinate at B, a point of the Line AB, it will be terminated at a certain geometrical Curve, whoſe Area, adjoining to the Abſciſs AB, is equal to the Area ADG.

18. And thus geometrical Figures may be found equal to other Figures, made by the application (in any Angle) of Arches of a Circle, of an Hyperbola, or of any other Curve, to the Sines right or verſed of thoſe Arches, or to any other right Lines that may be Geometrically determin'd.

19. As to Spirals, the matter will be very ſhort. For from the Center of Rotation A, the Arch DG being deſcribed, with any Radius AG, cutting the right Line AF in G, and the Spiral in D; ſince that Arch, as a Line moving upon the Abſciſs AG, deſcribes the Area of the Spiral AHDG, ſo that the Fluxion of that Area is to the Fluxion of the Rectangle 1 × AG, as the Arch GD to 1; if you raiſe the perpendicular right Line GL equal to that Arch, by moving in like manner upon the ſame Line AG, it will deſcribe the Area A*l*LG equal to the Area of the Spiral AHDG: The Curve A*l*L being a geometrical Curve. And further, if the Subtenſe AL be drawn, then $\triangle\, ALG = \frac{1}{2}AG \times GL = \frac{1}{2}AG \times GD =$ Sector AGD; therefore the complemental Segments AL*l* and ADH will alſo be equal. And this not only agrees to the Spiral of *Archimedes*, (in which caſe A*l*L becomes the Parabola of *Apollonius*,) but to any other whatever; ſo that all of them may be converted into equal geometrical Curves with the ſame eaſe.



20.

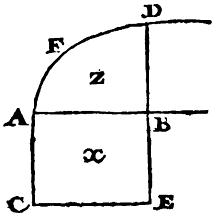
20. I might have produced more Specimens of the Conſtruction of this Problem, but theſe may ſuffice; as being ſo general, that whatever as yet has been found out concerning the Areas of Curves, or (I believe) can be found out, is in ſome manner contain'd herein, and is here determined for the moſt part with leſs trouble, and without the uſual perplexities.

21. But the chief uſe of this and the foregoing Problem is, that aſſuming the Conic Sections, or any other Curves of a known magnitude, other Curves may be found out that may be compared with theſe, and that their defining Equations may be diſpoſed orderly in a Catalogue or Table. And when ſuch a Table is conſtructed, when the Area of any Curve is to be found, if its defining Equation be either immediately found in the Table, or may be transformed into another that is contain'd in the Table, then its Area may be known. Moreover ſuch a Catalogue or Table may be apply'd to the determining of the Lengths of Curves, to the finding of their Centers of Gravity, their Solids generated by their rotation, the Superficies of thoſe Solids, and to the finding of any other flowing quantity produced by a Fluxion analogous to it.

PROB. IX.

To determine the Area of any Curve propoſed.

1. The reſolution of the Problem depends upon this, that from the relation of the Fluxions being given, the relation of the Fluents may be found, (as in Prob. 2.) And firſt, if the right Line BD, by the motion of which the Area required AFDB is deſcribed, move upright upon an Abſciſs AB given in poſition, conceive (as before) the Parallelogram ABEC to be deſcribed in the mean time on the other ſide AB, by a line equal to unity. And BE being ſuppos'd the Fluxion of the Parallelogram, BD will be the Fluxion of the Area required.



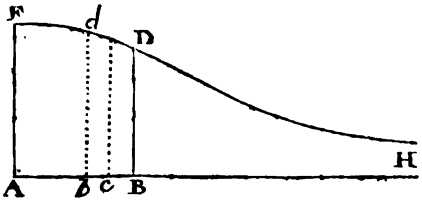
2. Therefore make AB $= x$, and then alſo ABEC $= 1 \times x = x$, and BE $= \dot{x}$. Call alſo the Area AFDB $= z$, and it will be BD $= \dot{z}$, as alſo $= \frac{\dot{z}}{\dot{x}}$, becauſe $\dot{x} = 1$. Therefore by the Equation expreſſing BD, at the ſame time the ratio of the Fluions $\frac{\dot{z}}{\dot{x}}$

is expreſs'd, and thence (by Prob. 2. Caſe 1.) may be found the relation of the flowing quantities x and z.

3. Ex. 1. When BD, or $\dot{z}$, is equal to ſome ſimple quantity.

4. Let there be given $\frac{xx}{a} = \dot{z}$, or $\frac{\dot{z}}{x}$, (the Equation to the Parabola,) and (Prob. 2.) there will ariſe $\frac{x^3}{3a} = z$. Therefore $\frac{x^3}{3a}$, or $\frac{1}{3}$ AB × BD, = Area of the Parabola AFDB.

5. Let there be given $\frac{x^3}{aa} = \dot{z}$, (an Equation to a Parabola of the ſecond kind,) and there will ariſe $\frac{x^4}{4a^2} = z$, that is, $\frac{1}{4}$ AB × BD = Area AFDB.



6. Let there be given $\frac{a^3}{xx} = \dot{z}$, or $a^3x^{-2} = \dot{z}$, (an Equation to an Hyperbola of the ſecond kind,) and there will ariſe $-a^3x^{-1} = z$, or $-\frac{a^3}{x} = z$. That is, AB × BD = Area HDBH, of an infinite length, lying on the other ſide of the Ordinate BD, as its negative value inſinuates.

7. And thus if there were given $\frac{a^4}{x^3} = \dot{z}$, there would ariſe $-\frac{a^4}{2xx} = z$.

8. Moreover, let $ax = \dot{z}\dot{z}$, or $a^{\frac{1}{2}}x^{\frac{1}{2}} = \dot{z}$, (an Equation again to the Parabola,) and there will ariſe $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} = z$, that is, $\frac{2}{3}$AB × BD = Area AFDB.

9. Let $\frac{a^3}{x} = \dot{z}\dot{z}$; then $-2a^{\frac{3}{2}}x^{\frac{1}{2}} = z$, or 2 AB × BD = AFDH.

10. Let $\frac{a^5}{x^3} = \dot{z}\dot{z}$; then $-\frac{2a^{\frac{5}{2}}}{x^{\frac{1}{2}}} = z$, or 2 AB × BD = HDBH.

11. Let $ax^2 = \dot{z}^3$; then $\frac{3}{5}a^{\frac{1}{3}}x^{\frac{5}{3}} = z$, or $\frac{3}{5}$ AB × BD = AFDH. And ſo in others.

12. Ex. 2. Where $\dot{z}$ is equal to an Aggregate of ſuch Quantities.

13. Let $x + \frac{xx}{a} = \dot{z}$; then $\frac{xx}{2} + \frac{xxx}{9a} = z$.

14. Let $a + \frac{a^3}{xx} = \dot{z}$; then $ax - \frac{a^3}{x} = z$.

15. Let $3x^{\frac{1}{2}} - \frac{5}{xx} + \frac{2}{x^{\frac{1}{2}}} = \dot{z}$; then $2x^{\frac{3}{2}} + \frac{5}{x} - 4x^{\frac{1}{2}} = z$.

16. Ex. 3. Where a previous reduction by Diviſion is required.

17. Let there be given $\frac{aa}{b+x}$, $= \dot{z}$ (an Equation to the *Apollonian* Hyperbola,) and the diviſion being performed *in infinitum*, it will be

$\dot{z} = \frac{aa}{b} - \frac{aax}{b^2} + \frac{aax^2}{b^3} - \frac{aax^3}{b^4}$, &c. And thence, (by Prob. 2.) as in the ſecond Set of Examples, you will obtain $z = \frac{a^2x}{b} - \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} - \frac{a^2x^4}{4b^4}$, &c.

18. Let there be given $\frac{1}{1+xx} = \dot{z}$, and by diviſion it will be $\dot{z} = 1 - x^2 + x^4 - x^6$, &c. or elſe $\dot{z} = \frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6}$, &c. And thence (by Prob. 2.) $z = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7$, &c. $=$ AFDB; or $z = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}$, &c. $=$ HDBH.

19. Let there be given $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x} = \dot{z}$, and by diviſion it will be $\dot{z} = 2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}}$, &c. And thence (by Prob. 2.) $z = \frac{4}{3}x^{\frac{3}{2}} - x^2 + \frac{14}{5}x^{\frac{5}{2}} - \frac{13}{3}x^3 + \frac{68}{7}x^{\frac{7}{2}}$, &c.

20. Ex. 4. Where a previous reduction is required by Extraction of Roots.

21. Let there be given $\dot{z} = \sqrt{aa + xx}$, (an Equation to the Hyperbola,) and the Root being extracted to an infinite multitude of terms, it will be $\dot{z} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{112a^7}$, &c. whence as in the foregoing $z = ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1008a^7}$, &c.

22. In the ſame manner if the Equation $\dot{z} = \sqrt{aa - xx}$ were given, (which is to the Circle,) there would be produced $z = ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1008a^7}$, &c.

23. And ſo if there were given $\dot{z} = \sqrt{x - xx}$, (an Equation alſo to the Circle,) by extracting the Root there would ariſe $\dot{z} = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$, &c. And therefore $z = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}}$, &c.

24. Thus $\dot{z} = \sqrt{aa + bx - xx}$, (an Equation again to the Circle,) by extraction of the Root it gives $\dot{z} = a + \frac{bx}{2a} - \frac{xx}{2a} - \frac{b^2x^2}{8a^3}$, &c. whence $z = ax + \frac{bx^2}{4a} - \frac{x^3}{6a} - \frac{b^2x^3}{24a^3}$, &c.

25. And thus $\sqrt{\frac{1 + axx}{1 - bxx}} = \dot{z}$, by a due reduction gives

$$\dot{z} = 1 \begin{array}{l} + \frac{1}{2}bx^2 \\ + \frac{1}{2}a \\ \end{array} \begin{array}{l} + \frac{3}{8}bbx^4, \\ + \frac{1}{4}ab \\ - \frac{1}{8}aa \end{array} \text{ \&c. then } z = x \begin{array}{l} + \frac{1}{6}bx^3 \\ + \frac{1}{6}a \\ \end{array} \begin{array}{l} + \frac{3}{40}bbx^5, \\ + \frac{1}{20}ab \\ - \frac{1}{40}aa \end{array} \text{ \&c.}$$

26.

26. Thus finally $\dot{z} = \sqrt[3]{a^3 + x^3}$, by the extraction of the Cubic Root, gives $\dot{z} = a + \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8}$, &c. and then (by Prob. 2.) $z = ax + \frac{x^4}{12a^2} - \frac{x^7}{63a^5} + \frac{x^{10}}{162a^8}$, &c. $=$ AFDB. Or elſe $\dot{z} = x + \frac{a^3}{3xx} - \frac{a^6}{9x^5} + \frac{5a^9}{81x^8}$, &c. And thence $z = \frac{x^2}{2} - \frac{a^3}{3x} + \frac{a^6}{36x^4} - \frac{5a^9}{567x^7}$, &c. $=$ HDBH.

27. Ex. 5. Where a previous reduction is required, by the reſolution of an affected Equation.

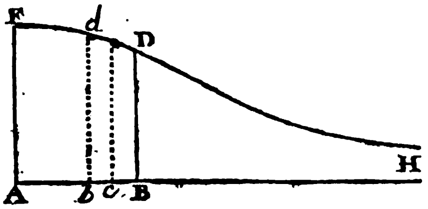
28. If a Curve be defined by this Equation $\dot{z}^3 + a^2\dot{z} + ax\dot{z} - 2a^3 - x^3 = 0$, extract the Root, and there will ariſe $\dot{z} = a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^4}{512aa}$, &c. whence will be obtain'd as before $z = ax - \frac{xx}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2}$, &c.

29. But if $\dot{z}^3 - c\dot{z}^2 - 2x^2\dot{z} - c^2\dot{z} + 2x^3 + c^3 = 0$ were the Equation to the Curve, the reſolution will afford a three-fold Root; either $\dot{z} = c + x - \frac{xx}{4c} + \frac{x^3}{32c^2}$, &c. or $\dot{z} = c - x + \frac{3x^2}{4c} - \frac{15x^3}{32cc}$, &c. or $\dot{z} = -c - \frac{x^2}{2c} - \frac{x^3}{2cc} + \frac{x^5}{4c^4}$, &c. And hence will ariſe the values of the three correſponding Areas, $z = cx + \frac{1}{2}x^2 - \frac{x^3}{12c} + \frac{x^4}{128c^2}$, &c. $z = cx - \frac{1}{2}x^2 + \frac{x^3}{4c} - \frac{15x^4}{128c^2}$, &c. and $z = -cx - \frac{x^3}{6c} - \frac{x^4}{8c^2} + \frac{x^6}{24c^4}$, &c.

30. I add nothing here concerning mechanical Curves, becauſe their reduction to the form of geometrical Curves will be taught afterwards.

31. But whereas the values of z thus found belong to Areas which are ſituate, ſometimes to a finite part AB of the Abſciſs, ſometimes to a part BH produced infinitely towards H, and ſometimes to both parts, according to their different terms: That the due value of the Area may be aſſign'd, adjacent to any portion of the Abſciſs, that Area is always to be made equal to the difference of the values of z, which belong to the parts of the Abſciſs, that are terminated at the beginning and end of the Area.

32. For Inſtance; to the Curve expreſs'd by the Equation $\frac{1}{1+xx} = z$,

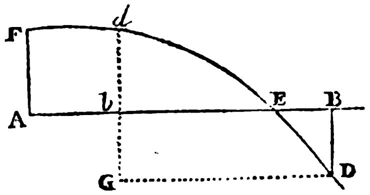


$=\dot{z}$, it is found that $z = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$, &c. Now that I may determine the quantity of the Area *bd*DB, adjacent to the part of the Abſciſs *b*B; from the value of *z* which ariſes by putting AB $=$ x, I take the value of *z* which ariſes by putting A*b* $= x$, and there remains $x - \frac{1}{3}x^3 + \frac{1}{5}x^5$, &c. $- x + \frac{1}{3}x^3 - \frac{1}{5}x^5$, &c. the value of that Area *bd*DB. Whence if A*b*, or *x*, be put equal to nothing, there will be had the whole Area AFDB $= x - \frac{1}{3}x^3 + \frac{1}{5}x^5$, &c.

33. To the ſame Curve there is alſo found $z = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}$, &c. Whence again, according to what is before, the Area *bd*DB $= \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5}$, &c. $- \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}$, &c. Therefore if AB, or x, be ſuppoſed infinite, the adjoining Area *bd*H toward H, which is alſo infinitely long, will be equivalent to $\frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5}$, &c. For the latter Series $- \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}$, &c. will vaniſh, becauſe of its infinite denominators.

34. To the Curve repreſented by the Equation $a + \frac{a^3}{xx} = \dot{z}$, it is found, that $z = ax - \frac{a^3}{x}$. Whence it is that $ax - \frac{a^3}{x} - ax + \frac{a^3}{x} =$ Area *bd*DB. But this becomes infinite, whether *x* be ſuppoſed nothing, or x infinite; and therefore each Area AFDB and *bd*H is infinitely great, and the intermediate parts alone, ſuch as *bd*DB, can be exhibited. And this always happens when the Abſciſs *x* is found as well in the numerators of ſome of the terms, as in the denominators of others, of the value of *z*. But when *x* is only found in the numerators, as in the firſt Example, the value of *z* belongs to the Area ſituate at AB, on this ſide the Ordinate. And when it is only in the denominators, as in the ſecond Example, that value, when the ſigns of all the terms are changed, belongs to the whole Area infinitely produced beyond the Ordinate.

35. If at any time the Curve-line cuts the Abſciſs, between the points *b* and B, ſuppoſe in E, inſtead of the Area will be had the difference *bd*E — BDE of the Areas at the different parts of the Abſciſs; to which if there be added the Rectangle BDG*b*, the Area *d*EDG will be obtain'd.



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36. But it is chiefly to be regarded, that when in the value of $\dot{z}$ any term is divided by x of only one dimenſion; the Area correſponding to that term belongs to the Conical Hyperbola; and therefore is to be exhibited by it ſelf, in an infinite Series: As is done in what follows.

37. Let $\frac{a^3 - a^2x}{ax + xx} = \dot{z}$ be an Equation to a Curve; and by diviſion it becomes $\dot{z} = \frac{aa}{x} - 2a + 2x - \frac{2x^2}{a} + \frac{2x^3}{aa}$, &c. and thence $z = \boxed{\frac{aa}{x}} - 2ax + x^2 - \frac{2x^3}{3a} + \frac{x^4}{2a^2}$; &c. And the Area *bd*DB $= \boxed{\frac{aa}{x}} - 2ax + x^2 - \frac{2x^3}{3a}$, &c. $- \boxed{\frac{aa}{x}} + 2ax - xx + \frac{2x^3}{3a}$, &c. Where by the Marks $\boxed{\frac{aa}{x}}$ and $\boxed{\frac{aa}{x}}$ I denote the little Areas belonging to the Terms $\frac{aa}{x}$ and $\frac{aa}{x}$.

38. Now that $\boxed{\frac{aa}{x}}$ and $\boxed{\frac{aa}{x}}$ may be found, I make A*b*, or x, to be definite, and *b*B indefinite, or a flowing Line, which therefore I call y; ſo that it will be $\boxed{\frac{aa}{x+y}} =$ to that Hyperbolical Area adjoining to *b*B, that is, $\boxed{\frac{aa}{x}} - \boxed{\frac{aa}{x}}$. But by Diviſion it will be $\frac{aa}{x+y} = \frac{aa}{x} - \frac{a^2y}{x^2} + \frac{a^2y^2}{x^3} - \frac{a^2y^3}{x^4}$, &c. and therefore, $\boxed{\frac{aa}{x+y}}$ or $\boxed{\frac{aa}{x}} - \boxed{\frac{aa}{x}} = \frac{a^2y}{x} - \frac{a^2y^2}{2x^2} + \frac{a^2y^3}{3x^3} - \frac{a^2y^4}{4x^4}$, &c. and therefore the whole Area required *bd*DB $= \frac{a^2y}{x} - \frac{a^2y^2}{2x^2} + \frac{a^2y^3}{3x^3}$, &c. $- 2ax + x^2 - \frac{2x^3}{3a}$, &c. $+ 2ax - xx + \frac{2x^3}{3a}$, &c.

39. After the ſame manner, AB, or x, might have been uſed for a definite Line, and then it would have been $\boxed{\frac{aa}{x}} - \boxed{\frac{aa}{x}} = \frac{a^2y^2}{x} + \frac{a^2y^2}{2x^2} + \frac{a^2y^3}{3x^3} + \frac{a^2y^4}{4x^4}$, &c.

40. Moreover, if *b*B be biſected in C, and AC be aſſumed to be of a definite length, and C*b* and CB indefinite; then making AC $= e$, and C*b* or CB $= y$, 'twill be $bd = \frac{aa}{e-y} = \frac{aa}{e} + \frac{a^2y}{e^2} + \frac{a^2y^2}{e^3} + \frac{a^2y^3}{e^4} + \frac{a^2y^4}{e^5}$; &c. and therefore the Hyperbolical Area adjacent

to the Part of the Abſciſs *b*C will be $\frac{a^2y}{e} + \frac{a^2y^2}{2e^2} + \frac{a^2y^3}{3e^3} + \frac{a^2y^4}{4e^4}$, &c. 'Twill be alſo DB $= \frac{aa}{e+y} = \frac{aa}{e} - \frac{aay}{e^2} + \frac{aay^2}{e^3} - \frac{aay^3}{e^4} + \frac{aay^4}{e^5}$, &c. And therefore the Area adjacent to the other part of the Abſciſs CB $= \frac{a^2y}{e} - \frac{a^2y^2}{2e^2} + \frac{a^2y^3}{3e^3} - \frac{a^2y^4}{4e^4} + \frac{a^2y^5}{5e^5}$, &c. And the Sum of theſe Areas $\frac{2a^2y}{e} + \frac{2a^2y^3}{3e^3} - \frac{2a^2y^5}{5e^5}$, &c. will be equivalent to $\boxed{\frac{aa}{x}} - \boxed{\frac{aa}{x}}$.

41. Thus in the Equation $\dot{z}^3 + \dot{z}^2 + \dot{z} - x^3 = 0$, denoting the nature of a Curve, its Root will be $\dot{z} = x - \frac{1}{3} - \frac{2}{9x} + \frac{7}{81xx} + \frac{5}{81x^3}$, &c. Whence there ariſes $z = \frac{1}{2}xx - \frac{1}{3}x - \boxed{\frac{2}{9x}} - \frac{7}{81x} - \frac{5}{162xx}$, &c. And the Area *bd*DB $= \frac{1}{2}x^2 - \frac{1}{3}x - \boxed{\frac{2}{9x}} - \frac{7}{81x}$, &c. $- \frac{1}{2}xx + \frac{1}{3}x + \boxed{\frac{2}{9x}} + \frac{7}{81x}$, &c. that is, $= \frac{1}{2}x^2 - \frac{1}{3}x - \frac{7}{81x}$, &c. $- \frac{1}{2}x^2 + \frac{1}{3}x + \frac{7}{81x}$, &c. $- \frac{4y}{9e} - \frac{4y^3}{27e^3} - \frac{4y^5}{45e^5}$, &c.

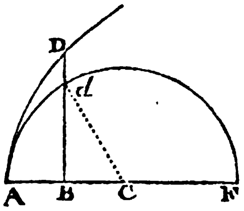
42. But this Hyperbolical term, for the moſt part, may be very commodiouſly avoided, by altering the beginning of the Abſciſs, that is, by increaſing or diminiſhing it by ſome given quantity. As in the former Example, where $\frac{a^3 - a^2x}{ax + xx} = \dot{z}$ was the Equation to the Curve, if I ſhould make *b* to be the beginning of the Abſciſs, and ſuppoſing A*b* to be of any determinate length $\frac{1}{2}a$, for the remainder of the Abſciſs *b*B, I ſhall now write x: That is, if I diminiſh the Abſciſs by $\frac{1}{2}a$, by writing $x + \frac{1}{2}a$ inſtead of x, it will become $\frac{\frac{1}{2}a^3 - a^2x}{\frac{3}{4}a^2 + 2ax + x^2} = \dot{z}$, and (by Diviſion) $\dot{z} = \frac{2}{3}a - \frac{28}{9}x + \frac{200x^2}{27a}$, &c. whence ariſes $z = \frac{2}{3}ax - \frac{14}{9}x^2 + \frac{200x^3}{81a}$, &c. $=$ Area *bd*DB.

43. And thus by aſſuming another and another point for the beginning of the Abſciſs, the Area of any Curve may be expreſs'd an infinite variety of ways.

44. Alſo the Equation $\frac{a^3 - a^2x}{ax + xx} = \dot{z}$ might have been reſolved into the two infinite Series $\dot{z} = \frac{a^3}{x^2} - \frac{a^4}{x^3} + \frac{a^5}{x^4}$, &c. $- a + x - \frac{xx}{a} + \frac{x^3}{a^2}$, &c. where there is found no Term divided by the firſt

Power of x. But ſuch kind of Series, where the Powers of x aſcend infinitely in the numerators of the one, and in the denominators of the other, are not ſo proper to derive the value of z from, by Arithmetical computation, when the Species are to be changed into Numbers.

45. Hardly any thing difficult can occur to any one, who is to undertake ſuch a computation in Numbers, after the value of the Area is obtain'd in Species. Yet for the more compleat illuſtration of the foregoing Doctrine, I ſhall add an Example or two.

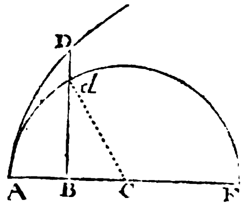


46. Let the Hyperbola AD be propoſed, whoſe Equation is $\sqrt{x+xx}=\dot{z}$; its Vertex being at A, and each of its Axes is equal to Unity. From what goes before, its Area ADB $=\frac{2}{3}x^{\frac{3}{2}}+\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}+\frac{1}{72}x^{\frac{9}{2}}-\frac{5}{704}x^{\frac{11}{2}}$, &c. that is $x^{\frac{1}{2}}$ into $\frac{2}{3}x+\frac{1}{5}x^2-\frac{1}{28}x^3+\frac{1}{72}x^4-\frac{5}{704}x^5$, &c. which Series may be infinitely produced by multiplying the laſt term continually by the ſucceeding terms of this Progreſſion $\frac{1.3}{2.5}x.$ $\frac{-1.5}{4.7}x.$ $\frac{-3.7}{6.9}x.$ $\frac{-5.9}{8.11}x.$ $\frac{-7.11}{10.13}x.$ &c. That is, the firſt term $\frac{2}{3}x^{\frac{3}{2}}\times\frac{1.3}{2.5}x$ makes the ſecond term $\frac{1}{5}x^{\frac{5}{2}}$: Which multiply'd by $\frac{-1.5}{4.7}x$ makes the third term $-\frac{1}{28}x^{\frac{7}{2}}$: Which multiply'd by $\frac{-3.7}{6.9}x$ makes $\frac{1}{72}x^{\frac{9}{2}}$ the fourth term; and ſo *ad infinitum*. Now let AB be aſſumed of any length, ſuppoſe $\frac{1}{4}$, and writing this Number for x, and its Root $\frac{1}{2}$ for $x^{\frac{1}{2}}$, and the firſt term $\frac{2}{3}x^{\frac{3}{2}}$ or $\frac{2}{3}\times\frac{1}{8}$, being reduced to a decimal Fraction, it becomes 0.083333333, &c. This into $\frac{1.3}{2.5.4}$ makes 0.00625 the ſecond term. This into $\frac{-1.5}{4.7.4}$ makes $-$ 0.0002790178, &c. the third term. And ſo on for ever. But the terms, which I thus deduce by degrees, I diſpoſe in two Tables; the affirmative terms in one, and the negative in another, and I add them up as you ſee here.

+ 0.0833333333333333	— 0.0002790178571429
625000000000	34679066051
271267361111	834465027
5135169396	26285354
144628917	961296
4954581	38676
190948	1663
7963	75
352	4
16	
1	— 0.0002825719389575
	+ 0.0896109885646518
+ 0.0896109885646518	0.0893284166257043

Then from the ſum of the Affirmatives I take the ſum of the negatives, and there remains 0.0893284166257043 for the quantity of the Hyperbolic Area ADB; which was to be found.

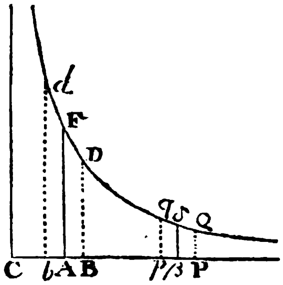
47. Now let the Circle A*d*F be propoſed, which is expreſſed by the equation $\sqrt{x-xx}=z$; that is, whoſe Diameter is unity, and from what goes before its Area A*d*B will be $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}}$, &c. In which Series, ſince the terms do not differ from the terms of the Series, which above expreſs'd the Hyperbolical Area, unleſs in the Signs + and —; nothing elſe remains to be done, than to connect the ſame numeral terms with other ſigns; that is, by ſubtracting the connected ſums of both the afore-mention'd tables, 0.0898935605036193 from the firſt term doubled 0.1666666666666, &c. and the remainder 0.0767731061630473 will be the portion A*d*B of the circular Area, ſuppoſing AB to be a fourth part of the diameter. And hence we may obſerve, that tho' the Areas of the Circle and Hyperbola are not compared in a Geometrical conſideration, yet each of them is diſcover'd by the ſame Arithmetical computation.



48. The portion of the circle A*d*B being found, from thence the whole Area may be derived. For the Radius *d*C being drawn, multiply B*d*, or $\frac{1}{4}\sqrt{3}$, into BC, or $\frac{1}{4}$, and half of the product $\frac{1}{32}\sqrt{3}$, or 0.0541265877365275 will be the value of the Triangle C*d*B; which added to the Area A*d*B, there will be had the Sector AC*d* = 0.1308996938995747, the ſextuple of which 0.7853981633974482 is the whole Area.

49. And hence by the way the length of the Circumference will be 3.1415926535897928, by dividing the Area by a fourth part of the Diameter.

50. To theſe we ſhall add the calculation of the Area comprehended between the Hyperbola *d*FD and its Aſymptote CA. Let C be the Center of the Hyperbola, and putting CA $= a$, AF $= b$, and AB $=$ A*b* $= x$; 'twill be $\frac{ab}{a+x} =$ BD, and $\frac{ab}{a-x} = bd$; whence the Area AFDB $= bx - \frac{bxx}{2a} + \frac{bx^3}{3a^2} - \frac{bx^4}{4a^3}$, &c. and the Area AF*db* $= bx + \frac{bx^2}{2a} + \frac{bx^3}{3a^2} + \frac{bx^4}{4a^3}$, &c. and the ſum *bd*DB $= 2bx + \frac{2bx^3}{3a^2} + \frac{2bx^5}{5a^4} + \frac{2bx^7}{7a^6}$, &c. Now let us ſuppoſe CA $=$ AF $= 1$, and A*b* or AB $= \frac{1}{10}$, C*b* being 0.9, and CB $= 1.1$; and ſubſtituting theſe numbers for *a*, *b*, and *x*, the firſt term of the Series becomes 0.2, the ſecond 0.0006666666, &c. the third 0.000004; and ſo on, as you ſee in this Table.



0.2000000000000000
6666666666666
40000000000
285714286
2222222
18182
154
1

The ſum 0.2006706954621511 $=$ Area *bd*DB.

51. If the parts of this Area A*d* and AD be deſired ſeparately, ſubtract the leſſer BA from the greater *d*A, and there will remain $\frac{bx^2}{a} + \frac{bx^4}{2a^3} + \frac{bx^6}{3a^5} + \frac{bx^8}{4a^7}$, &c. Where if 1 be wrote for *a* and *b*, and $\frac{1}{10}$ for *x*, the terms being reduced to decimals will ſtand thus;

0.0100000000000000
500000000000
3333333333
25000000
200000
1667
14

The ſum 0.0100503358535014 $=$ A*d* — AD.

52. Now if this difference of the Areas be added to, and ſubtracted from, their ſum before found, half the aggregate 0.1053605156578263 will be the greater Area Ad, and half of the remainder 0.0953101798043248 will be the leſſer Area AD.

53. By the ſame tables thoſe Areas AD and Ad will be obtain'd alſo, when AB and Ab are ſuppos'd $\frac{1}{100}$, or $CB = 1.01$, and $Cb = 0.99$, if the numbers are but duly transferr'd to lower places, as may be here ſeen.

0.0200000000000000
6666666666
400000
28
Sum 0 0200006667066695 $= bD$.

0.0010000000000000
50000000
3333
Sum 0.0010000050003333 $Ad - AD$.

$\frac{1}{2}$ Aggr. 0.0100503358535014 $= Ad$, and $\frac{1}{2}$ Reſid. 0.0099503308531681 $= AD$.

54. And ſo putting AB and $Ab = \frac{1}{1000}$, or $CB = 1.001$, and $Cb = 0.999$, there will be obtain'd $Ad = 0.0010005003335835$; and $AD = 0.0009995003330835$.

55. In the ſame manner (if CA and $AF = 1$) putting AB and $Ab = 0.2$, or 0.02, or 0.002, theſe Areas will ariſe,

$Ad = 0.2231435513142097$, and $AD = 0.1823215567939546$,
or $Ad = 0.0202027073175194$, and $AD = 0.0198026272961797$,
or $Ad = 0.002002$ and $AD = 0.001$

56. From theſe Areas thus found it will be eaſy to derive others; by addition and ſubtraction alone: For as it is $\frac{1.2}{0.8}$ into $\frac{1.2}{0.9} = 2$, the ſum of the Areas 0.6931471805599453 belonging to the Ratio's $\frac{1.2}{0.8}$ and $\frac{1.2}{0.9}$, (that is, inſiſting upon the parts of the Abſciſs $1.2 - 0.8$ and $1.2 - 0.9$,) will be the Area $AF\delta\beta$, $C\beta$ being $= 2$, as is known. Again, ſince $\frac{1.2}{0.8}$ into $2 = 3$, the ſum 1.0986122886681097 of the Area's belonging to $\frac{1.2}{0.8}$ and 2, will be the Area $AF\delta\beta$, $C\beta$ being 3. Again, as it is $\frac{2 \times 2}{0.8} = 5$, and $2 \times 5 = 10$, by a due addition of Areas will be obtain'd $1.6093379124341004 = AF\delta\beta$, when $C\beta = 5$; and $2.302585092994045 7 = AF\delta\beta$, when $C\beta = 10$. And thus, ſince $10 \times 10 = 100$, and $10 \times 100 = 1000$, and $\sqrt{5} \times 10 \times 0.98 = 7$, and $10 \times 1.1 = 11$, and $\frac{1000 \times 1.001}{7 \times 11} = 13$, and $\frac{100 \times 0.998}{2} = 499$; it is plain, that the Area $AF\delta\beta$ may be found by the compoſition of the Areas found before, when $C\beta = 100$; 1000;

7;

7; or any other of the above-mention'd numbers, AB = BF being ſtill unity. This I was willing to inſinuate, that a method might be derived from hence, very proper for the conſtruction of a Canon of Logarithms, which determines the Hyperbolical Areas, (from which the Logarithms may eaſily be derived,) correſponding to ſo many Prime numbers, as it were by two operations only, which are not very troubleſome. But whereas that Canon ſeems to be derivable from this fountain more commodiouſly than from any other, what if I ſhould point out its conſtruction here, to compleat the whole?

57. Firſt therefore having aſſumed 0 for the Logarithm of the number 1, and 1 for the Logarithm of the number 10, as is generally done, the Logarithms of the Prime numbers 2, 3, 5, 7, 11, 13, 17, 37, are to be inveſtigated, by dividing the Hyperbolical Areas now found by 2.302585092994057, which is the Area correſponding to the number 10: Or which is the ſame thing, by multiplying by its reciprocal 0.4342944819032518. Thus for Inſtance, if 0.69314718, &c. the Area correſponding to the number 2, were multiply'd by 0.43429, &c. it makes 0.3010299956639812 the Logarithm of the number 2.

58. Then the Logarithms of all the numbers in the Canon, which are made by the multiplication of theſe, are to be found by the addition of their Logarithms, as is uſual. And the void places are to be interpolated afterwards, by the help of this Theorem.

59. Let n be a Number to which a Logarithm is to be adapted, x the difference between that and the two neareſt numbers equally diſtant on each ſide, whoſe Logarithms are already found, and let d be half the difference of the Logarithms. Then the required Logarithm of the Number n will be obtain'd by adding $d + \frac{dx}{2n} + \frac{dx^3}{12n^3}$, &c. to the Logarithm of the leſſer number. For if the numbers are expounded by Cp, Cβ, and CP, the rectangle CBD or C$\beta\delta = 1$, as before, and the Ordinates pq and PQ being raiſed; if n be wrote for Cβ, and x for βp or βP, the Area pqQP or $\frac{2x}{n} + \frac{2x^3}{3n^3} + \frac{2x^5}{5n^5}$, &c. will be to the Area $pq\delta\beta$ or $\frac{x}{n} + \frac{x^2}{2n^2} + \frac{x^3}{3n^3}$, &c. as the difference between the Logarithms of the extream numbers or $2d$, to the difference between the Logarithms of the leſſer and of the middle

O one;

one; which therefore will be $\frac{\frac{dx}{n} + \frac{dx^2}{2n^2} + \frac{dx^3}{3n^3} \text{ \&c.}}{\frac{x}{n} + \frac{x^3}{3n^3} + \frac{x^5}{5n^5} \text{ \&c.}}$, that is, when the diviſion is perform'd, $d + \frac{dx}{2n} + \frac{dx^3}{12n^3}$ &c.

60. The two firſt terms of this Series $d + \frac{dx}{2n}$ I think to be accurate enough for the conſtruction of a Canon of Logarithms, even tho' they were to be produced to fourteen or fifteen figures; provided the number, whoſe Logarithm is to be found, be not leſs than 1000. And this can give little trouble in the calculation, becauſe x is generally an unit, or the number 2. Yet it is not neceſſary to interpolate all the places by the help of this Rule. For the Logarithms of numbers which are produced by the multiplication or diviſion of the number laſt found, may be obtain'd by the numbers whoſe Logarithms were had before, by the addition or ſubtraction of their Logarithms. Moreover by the differences of the Logarithms, and by their ſecond and third differences, if there be occaſion, the void places may be more expeditiouſly ſupply'd; the foregoing Rule being to be apply'd only, when the continuation of ſome full places is wanted, in order to obtain thoſe differences.

61. By the ſame method rules may be found for the intercalation of Logarithms, when of three numbers the Logarithms of the leſſer and of the middle number are given, or of the middle number and of the greater; and this although the numbers ſhould not be in Arithmetical progreſſion.

62. Alſo by purſuing the ſteps of this method, rules might be eaſily diſcover'd, for the conſtruction of the tables of artificial Sines and Tangents, without the aſſiſtance of the natural Tables. But of theſe things only by the bye.

63. Hitherto we have treated of the Quadrature of Curves, which are expreſs'd by Equations conſiſting of complicate terms; and that by means of their reduction to Equations, which conſiſt of an infinite number of ſimple terms. But whereas ſuch Curves may ſometimes be ſquared by finite Equations alſo, or however may be compared with other Curves, whoſe Areas in a manner may be conſider'd as known; of which kind are the Conic Sections: For this reaſon I thought fit to adjoin the two following catalogues or tables of Theorems, according to my promiſe, conſtructed by the help of the *7th* and *8th* aforegoing Propoſitions.

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64. The firſt of theſe exhibits the Areas of ſuch Curves as can be ſquared; and the ſecond contains ſuch Curves, whoſe Areas may be compared with the Areas of the Conic Sections. In each of theſe, the letters d, e, f, g, and h, denote any given quantities, x and z the Abſciſſes of Curves, v and y parallel Ordinates, and s and t Areas, as before. The letters η and θ, annex'd to the quantity z, denote the number of the dimenſions of the ſame z, whether it be integer or fractional, affirmative or negative. As if $\eta = 3$, then $z^{\eta} = z^{3}$, $z^{2\eta} = z^{6}$, $z^{-\eta} = z^{-3}$ or $\frac{1}{z^{3}}$, $z^{\eta+1} = z^{4}$, and $z^{\eta-1} = z^{2}$.

65. Moreover in the values of the Areas, for the ſake of brevity, is written R inſtead of this Radical $\sqrt{e+fz^{\eta}}$, or $\sqrt{e+fz^{\eta}+gz^{2\eta}}$, and p inſtead of $\sqrt{h+iz^{\eta}}$, by which the value of the Ordinate y is affected.

66. *The firſt Table, of ſome Curvilinear Areas related to Rectilinear Figures, conſtructed by Prob. 7.*

Order of Curves.		Values of their Areas.
I.	$dz^{\eta-1}=y$	$\frac{d}{\eta}z^{\eta}=t.$
II.	$\frac{dz^{\eta-1}}{ee+2efz^{\eta}+ffz^{2\eta}}=y$	$\frac{dz^{\eta}}{\eta e^{2}+\eta efz^{\eta}}=t$, or $\frac{-d}{\eta ef+\eta ffz^{\eta}}=t.$
III. 1	$dz^{\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{2d}{3\eta f}R^{3}=t.$
2	$dz^{2\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{-4e+6fz^{\eta}}{15\eta ff}dR^{3}=t.$
3	$dz^{3\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{16ee-24efz^{\eta}+30ffz^{2\eta}}{105\eta f^{3}}dR^{3}=t.$
4	$dz^{4\eta-1}\sqrt{e+fz^{\eta}}=y$	$\frac{-96e^{3}+144e^{2}fz^{\eta}-180ef^{2}z^{2\eta}+210f^{3}z^{3\eta}}{945\eta f^{4}}dR^{3}=t.$
IV. 1	$\frac{dz^{\eta-1}}{\sqrt{e+fz^{\eta}}}=y$	$\frac{2d}{\eta f}R=t.$
2	$\frac{dz^{2\eta-1}}{\sqrt{e+fz^{\eta}}}=y$	$\frac{-4e+2fz^{\eta}}{3\eta ff}dR=t.$
3	$\frac{dz^{3\eta-1}}{\sqrt{e+fz^{\eta}}}=y$	$\frac{16e^{2}-8efz^{\eta}+6ffz^{2\eta}}{15\eta f^{3}}dR=t.$
4	$\frac{dz^{4\eta-1}}{\sqrt{e+fz^{\eta}}}=y$	$\frac{-96e^{3}+48e^{2}fz^{\eta}-36ef^{2}z^{2\eta}+30f^{3}z^{3\eta}}{105\eta f^{4}}dR=t.$

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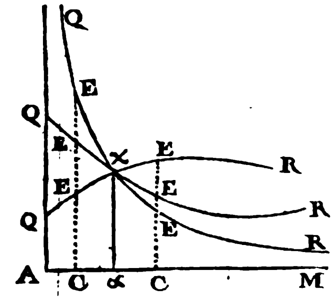
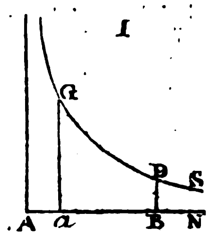
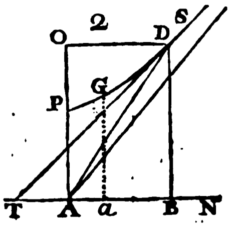
		Order of Curves.	Values of Areas.
V.	1	$2\theta e z^{\theta-1} \begin{array}{l}+2\theta\\+3n\end{array} f z^{\theta+n-1}$ into $\frac{1}{2}\sqrt{e+fz^n} = y.$	$z^\theta R^3 = t.$
	2	$2\theta e z^{\theta-1} \begin{array}{l}+2\theta\\+3n\end{array} f z^{\theta+n-1} \begin{array}{l}+2\theta\\+6n\end{array} g z^{\theta+2n-1}$ into $\frac{1}{2}\sqrt{e+fz^n+gz^{2n}} = y.$	$z^\theta R^3 = t.$
VI.	1	$\frac{2\theta e z^{\theta-1} + \overline{2\theta+n}\times f z^{\theta+n-1}}{2\sqrt{e+fz^n}} = y.$	$z^\theta R = t.$
	2	$\frac{2\theta e z^{\theta-1} + \overline{2\theta+n}\times f z^{\theta+n-1} + \overline{2\theta+2n}\times g z^{\theta+2n-1}}{2\sqrt{e+fz^n+gz^{2n}}} = y.$	$z^\theta R = t.$
VII.	1	$\frac{2\theta e z^{\theta-1} + \overline{2\theta-n}\times f z^{\theta+n-1}}{e+fz^n \text{ into } 2\sqrt{e+fz^n}} = y.$	$\frac{z^\theta}{R} = t.$
	2	$\frac{2\theta e z^{\theta-1} + \overline{2\theta-n}\times f z^{\theta+n-1} + \overline{2\theta-2n}\times g z^{\theta+2n-1}}{e+fz^n+gz^{2n} \text{ into } 2\sqrt{e+fz^n+gz^{2n}}} = y.$	$\frac{z^\theta}{R} = t.$
VIII.	1	$\frac{2\theta e z^{\theta-1} + \overline{2\theta-2n}\times f z^{\theta+n-1}}{ee+2efz^n+ffz^{2n}} = 2y.$	$\frac{z^\theta}{R^2}$ (or $\frac{z^\theta}{e+fz^n}$) $= t.$
	2	$\frac{2\theta e z^{\theta-1} + \overline{2\theta-2n}\times f z^{\theta+n-1} + \overline{2\theta-4n}\times g z^{\theta+2n-1}}{e^2+2efz^n+\overline{ff+2eg}\times z^{2n}+2fgz^{3n}+ggz^{4n}} = 2y.$	$\frac{z^\theta}{R^2}$ (or $\frac{z^\theta}{e+fz^n+gz^{2n}}$) $= t.$
IX.		$2\theta e b z^{\theta-1} \begin{array}{l}+\overline{2\theta+3n}\times fb\\+\overline{2\theta+n}\times ei\end{array} z^{\theta+n-1} + \overline{2\theta+4n}\times fiz^{\theta+2n-1}$ into $\frac{\sqrt{e+fz^n}}{2\sqrt{b+iz^n}} = y.$	$z^\theta R^3 p = t.$
X.		$2\theta e b z^{\theta-1} \begin{array}{l}+\overline{2\theta+3n}\times fb\\+\overline{2\theta-n}\times ei\end{array} z^{\theta+n-1} + \overline{2\theta+2n}\times fiz^{\theta+2n-1}$ into $\frac{\sqrt{e+fz^n}}{b+iz^n \text{ into } 2\sqrt{b+iz^n}} = y.$	$\frac{z^\theta R^3}{p} = t.$

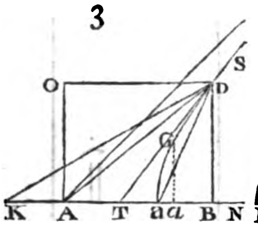
67. Other things of the ſame kind might have been added; but I ſhall now paſs on to another ſort of Curves, which may be compared with the Conic Sections. And in this Table or Catalogue you have the propoſed Curve repreſented by the Line $QE\chi R$, the beginning of whoſe Abſciſs is A, the Abſciſs AC, the Ordinate CE, the beginning of the Area $\alpha\chi$, and the Area deſcribed $\alpha\chi EC$. But the beginning of this Area, or the initial term, (which commonly either commences at the beginning of the Abſciſs A, or recedes to an infinite diſtance,) is found by ſeeking the length of the Abſciſs $A\alpha$, when the value of the Area is nothing, and by erecting the perpendicular $\alpha\chi$.

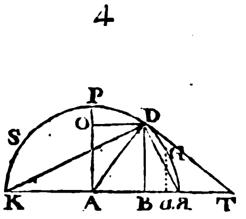


68. After the ſame manner you have the Conic Section repreſented by the Line PDG, whoſe Center is A, Vertex a, rectangular Semidiameters Aa and AP, the beginning of the Abſciſs A, or a, or α, the Abſciſs AB, or aB, or αB, the Ordinate BD, the Tangent DT meeting AB in T, the Subtenſe aD, and the Rectangle inſcribed or adſcribed ABDO.









69. Therefore retaining the letters before defined, it will be $AC = z$, $CE = y$, $\alpha\chi EC = t$, AB or $aB = x$, $BD = v$, and ABDP or $aGDB = s$. And beſides, when two Conic Sections are required, for the determination of any Area, the Area of the latter ſhall be call'd σ, the Abſciſs ξ, and the Ordinate Υ. Put p for $\sqrt{ff - 4eg}$.

70. *The ſecond Table, of ſome Curvilinear Areas, related to the Conic Sections, conſtructed by the help of Prob. 8.*

Forms of Curves.			Conic Sections.		Values of the Areas.	
			Abſciſs.	Ordinate.		
I.	1	$\frac{dz^{n-1}}{e+fz^n}=y$	$z^n=x$	$\frac{d}{e+fx}=v$	$\frac{1}{n}s=t=\frac{a\mathrm{GDB}}{n}.$	*Fig.* 1.
	2	$\frac{dz^{2n-1}}{e+fz^n}=y$	$z^n=x$	$\frac{d}{e+fx}=v$	$\frac{d}{nf}z^n-\frac{e}{nf}s=t.$	
	3	$\frac{dz^{3n-1}}{e+fz^n}=y$	$z^n=x$	$\frac{d}{e+fx}=v$	$\frac{d}{2nf}z^{2n}-\frac{de}{nf^2}z^n+\frac{e^2}{nf^2}s=t.$	
II.	1	$\frac{dz^{\frac{1}{2}n-1}}{e+fz^n}=y$	$\sqrt{\frac{d}{e+fz^n}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$	$\frac{2xv\div 4s}{n}=t=\frac{4}{n}\mathrm{ADG}a.$	*Fig.* 3, 4.
	2	$\frac{dz^{\frac{3}{2}n-1}}{e+fz^n}=y$	$\sqrt{\frac{d}{e+fz^n}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$	$\frac{2d}{nf}z^{\frac{1}{2}n}+\frac{4es-2exv}{nf}=t.$	
	3	$\frac{dz^{\frac{5}{2}n-1}}{e+fz^n}=y$	$\sqrt{\frac{d}{e+fz^n}}=x$	$\sqrt{\frac{d}{f}-\frac{e}{f}x^2}=v$	$\frac{2d}{3nf}z^{\frac{3}{2}n}-\frac{2de}{nf^2}z^{\frac{1}{2}n}+\frac{2e^2xv-4e^2s}{nf^2}=t.$	
III.	1	$\frac{d}{z}\sqrt{e+fz^n}=y.$	$\frac{1}{z^n}=x^2$	$\sqrt{f+ex^2}=v$	$\frac{4de}{nf}\times\overline{\frac{v^3}{2ex}-s}=t=\frac{4de}{nf}$ into aGDT, or into APDB $\div$ TDB.	[*Fig.* 2, 3, 4.
		or thus	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{8de^2}{nf^2}\times\overline{s-\frac{1}{2}xv-\frac{fv}{4e}+\frac{f^2v}{4e^2x}}=t=\frac{8de^2}{nf^2}$ into aGDA $+\frac{f^2v}{4e^2x}$	[*Fig.* 3, 4.
	2	$\frac{d}{z^{n+1}}\sqrt{e+fz^n}=y$	$\frac{1}{z^n}=x^2$	$\sqrt{f+ex^2}=v$	$-\frac{2d}{n}s=t=\frac{2d}{n}$ APDB or $\frac{2d}{n}$ aGDB,	*Fig.* 2, 3, 4.
		or thus	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{4de}{nf}\times\overline{s-\frac{1}{2}xv-\frac{fv}{2e}}=t=\frac{4de}{nf}\times$ aGDK.	*Fig.* 3, 4.
	3	$\frac{d}{z^{2n+1}}\sqrt{e+fz^n}=y$	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$-\frac{d}{n}s=t=\frac{d}{n}\times-$ aGDB or BDPK.	*Fig.* 4.
	4	$\frac{d}{z^{3n+1}}\sqrt{e+fz^n}=y$	$\frac{1}{z^n}=x$	$\sqrt{fx+ex^2}=v$	$\frac{3dfs-2dv^3}{6ne}=t.$	

		Forms of Curves.	Conic Sections.		Values of the Areas.
			Abſciſs.	Ordinate.	
IV.	1	$\frac{d}{z\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x^{2}$	$\sqrt{f+ex^{2}}=v$	$\frac{4d}{\eta f}\times\overline{\tfrac{1}{2}xv-s}=t=\frac{4d}{\eta f}$ into PAD or into aGDA. *Fig.* 2, 3, 4
		or thus	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^{2}}=v$	$\frac{8de}{\eta f^{2}}\times\overline{s-\tfrac{1}{2}xv-\frac{fv}{4e}}=t=\frac{8de}{\eta f^{2}}$ into aGDA. *Fig.* 3, 4.
	2	$\frac{d}{z^{\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x^{2}$	$\sqrt{f+ex^{2}}=v$	$\frac{2d}{\eta e}\times\overline{s-xv}=t=\frac{2d}{\eta e}$ into POD or into AODGa. *Fig.* 2, 3, 4
		or thus	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^{2}}=v$	$\frac{4d}{\eta f}\times\overline{\tfrac{1}{2}xv-s}=t=\frac{4d}{\eta f}$ into aDG*a*. *Fig.* 3, 4
	3	$\frac{d}{z^{2\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^{2}}=v$	$\frac{d}{\eta e}\times\overline{3s-2xv}=t=\frac{d}{\eta e}$ into 3aDGa − △aDB. *Fig.* 3, 4
	4	$\frac{d}{z^{3\eta+1}\sqrt{e+fz^{\eta}}}=y$	$\frac{1}{z^{\eta}}=x$	$\sqrt{fx+ex^{2}}=v$	$\frac{10dfxv-15dfs-2dex^{2}v}{6\eta e}=t.$
V.	1	$\frac{dz^{\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}+gz^{2\eta}}}=x$	$\sqrt{\frac{d}{g}+\frac{f^{2}-4eg}{4g^{2}}x^{2}}=v$	$\frac{xv-2s}{\eta}=t.$
		or thus	$\sqrt{\frac{dz^{2\eta}}{e+fz^{\eta}+gz^{2\eta}}}=x$	$\sqrt{\frac{d}{e}+\frac{f^{2}-4eg}{4e^{2}}x^{2}}=v$	$\frac{9s-xv}{\eta}=t.$
	2	$\frac{dz^{2\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y$	$\sqrt{\frac{d}{e+fz^{\eta}+gz^{2\eta}}}=x$ $fz^{\eta}+gz^{2\eta}=\xi$	$\sqrt{\frac{d}{g}+\frac{f^{2}-4eg}{4g^{2}}x^{2}}=v$ $\frac{1}{e+\xi}=\Upsilon$	$\frac{d\sigma+2fs-fxv}{2\eta g}=t.$
VI.	1	$\frac{dz^{\frac{1}{2}\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y$	$\sqrt{\frac{2dg}{f-p+2gz^{\eta}}}=x$ $\sqrt{\frac{2dg}{f+p+2gz^{\eta}}}=\xi$	$\sqrt{d+\frac{-f+p}{2g}x^{2}}=v$ $\sqrt{d+\frac{-f-p}{2g}\xi^{2}}=\Upsilon$	$\frac{2xv-4s-2\xi\Upsilon+4\sigma}{\eta p}=t.$
	2	$\frac{dz^{\frac{3}{2}\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y$	$\sqrt{\frac{2dez^{\eta}}{fz^{\eta}-pz^{\eta}+2e}}=x$ $\sqrt{\frac{2dez^{\eta}}{fz^{\eta}+pz^{\eta}+2e}}=\xi$	$\sqrt{d+\frac{-f+p}{2e}x^{2}}=v$ $\sqrt{d+\frac{-f-p}{2e}\xi^{2}}=\Upsilon$	$\frac{4s-2xv-4\sigma+2\xi\Upsilon}{\eta p}=t.$

		Forms of Curves.	Conic Sections.		Values of the Areas.
			Abſciſs.	Ordinate.	
VII.	1	$\frac{d}{z}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$	$z^{\eta}=x$ $\frac{1}{z^{\eta}}=\xi$	$\sqrt{e+fx+gx^2}=v$ $\sqrt{g+f\xi+e\xi^2}=\Upsilon$	$\frac{4de^2\xi\Upsilon+2def\Upsilon-2dfgxv\begin{smallmatrix}+4deg\\-2ffd\end{smallmatrix}v-8de^2\sigma+4dfgs}{4neg-nff}=t.$
	2	$dz^{\eta-1}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{d}{n}s=t=\frac{d}{n}\times\alpha\mathrm{GDB}.$ *Fig.* 2, 3, 4.
	3	$dz^{2\eta-1}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{d}{3ng}v^3-\frac{df}{2ng}s=t.$
	4	$dz^{3\eta-1}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{6dgx-5df}{24ng^2}v^3+\frac{5df^2-4deg}{16ng^2}s=t.$
VIII.	1	$\frac{dz^{\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{8dgs-4dgxv-2dfv}{4neg-nf^2}=t=\frac{8dg}{4neg-nf^2}\times\alpha\mathrm{GDB}\pm\Delta\mathrm{DBA}.$ [*Fig.* 2, 4.
	2	$\frac{dz^{2\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{-4dfs+2dfxv+4dev}{4neg-nf^2}=t.$
	3	$\frac{dz^{3\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{\begin{smallmatrix}3dff\\-4deg\end{smallmatrix}s\begin{smallmatrix}-2dff\\+4deg\end{smallmatrix}xv-2defv}{4neg^2-nf^2g}=t.$
	4	$\frac{dz^{4\eta-1}}{\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y$	$z^{\eta}=x$	$\sqrt{e+fx+gx^2}=v$	$\frac{\begin{smallmatrix}36defg\\-15df^3\end{smallmatrix}s\begin{smallmatrix}+8degg\\-2df^2g\end{smallmatrix}x^2v\begin{smallmatrix}+10dfff\\-28defg\end{smallmatrix}xv\begin{smallmatrix}+10deff\\-16de^2g\end{smallmatrix}v}{24neg^2-6nf^2g^2}=t.$
IX.	1	$\frac{dz^{\eta-1}\sqrt{e+fz^{\eta}}}{g+hz^{\eta}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$	$\frac{\begin{smallmatrix}4fg\\-4eh\end{smallmatrix}s\begin{smallmatrix}-2fg\\+2eh\end{smallmatrix}xv+2df\frac{v}{x}}{nfh}=t$
	2	$\frac{dz^{2\eta-1}\sqrt{e+fz^{\eta}}}{g+hz^{\eta}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$	$\frac{\begin{smallmatrix}4egh\\-4fgg\end{smallmatrix}s\begin{smallmatrix}-2egh\\+2fgg\end{smallmatrix}xv+\frac{2}{3}dh\frac{v^3}{x^3}-2dfg\frac{v}{x}}{nfh^2}=t.$

		Forms of Curves.	Conic Sections. Abſciſs.	Ordinate.	Values of the Areas.
X.	1	$\frac{dz^{\eta-1}}{g+hz^{\eta}\sqrt{e+fz^{\eta}}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$	$\frac{2xv-4s}{nf}=t=\frac{4}{nf}$ ADGa. Fig. 3, 4.
	2	$\frac{dz^{2\eta-1}}{g+hz^{\eta}\sqrt{e+fz^{\eta}}}=y$	$\sqrt{\frac{d}{g+hz^{\eta}}}=x$	$\sqrt{\frac{df}{h}+\frac{eh-fg}{h}x^2}=v$	$\frac{4gs-2gxv+2d\frac{v}{x}}{nfh}=t.$
XI.	1	$dz^{-1}\sqrt{\frac{e+fz^{\eta}}{g+hz^{\eta}}}=y$	$\sqrt{g+hz^{\eta}}=x$ $\sqrt{h+gz^{-\eta}}=\xi$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}=v$ $\sqrt{\frac{fg-eh}{g}+\frac{e}{g}\xi^2}=\Upsilon$	$\frac{2dxv^3z^{-\eta}-4dfs-4dea}{nfg-neh}=t.$
	2	$dz^{\eta-1}\sqrt{\frac{e+fz^{\eta}}{g+hz^{\eta}}}=y$	$\sqrt{g+hz^{\eta}}=x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}=v$	$\frac{2d}{nh}s=t.$
	3	$dz^{2\eta-1}\sqrt{\frac{e+fz^{\eta}}{g+hz^{\eta}}}=y$	$\sqrt{g+hz^{\eta}}=x$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}x^2}=v$	$\frac{dhxv^3\begin{smallmatrix}-3dfg\\-deh\end{smallmatrix}s}{2nf}=t.$

71. Before I go on to illuſtrate by Examples the Theorems that are deliver'd in theſe claſſes of Curves, I think it proper to obſerve,

72. I. That whereas in the Equations repreſenting Curves, I have all along ſuppoſed all the ſigns of the quantities d, e, f, g, h, and i to be affirmative; whenever it ſhall happen that they are negative, they muſt be changed in the ſubſequent values of the Abſciſs and Ordinate of the Conic Section, and alſo of the Area required.

73. II. Alſo the ſigns of the numeral Symbols η and θ, when they are negative, muſt be changed in the values of the Areas. Moreover their Signs being changed, the Theorems themſelves may acquire a new form. Thus in the 4th Form of Table 2, the Sign of η being changed, the 3d Theorem becomes $\frac{d}{z^{-2\eta+1}\sqrt{e+fz^{-\eta}}} = y$, $\frac{1}{z^{-\eta}} = x$, &c. that is, $\frac{dz^{3\eta-1}}{\sqrt{ez^{2\eta}+fz^{\eta}}} = y$, $z^{\eta} = x$, $\sqrt{fx+ex^2} = v$, $\frac{d}{\eta e}$ into $2xv - 3s = t$. And the ſame is to be obſerved in others.

74. III. The ſeries of each order, excepting the 2d of the 1ſt Table, may be continued each way *ad infinitum.* For in the Series of the 3d and 4th Order of Table 1, the numeral co-efficients of the initial terms, (2, −4, 16, −96, 768, &c.) are form'd by multiplying the numbers −2, −4, −6, −8, −10, &c. continually into each other; and the co-efficients of the ſubſequent terms are derived from the initials in the 3d Order, by multiplying gradually by $-\frac{3}{2}$, $-\frac{5}{4}$, $-\frac{7}{6}$, $-\frac{9}{8}$, $-\frac{11}{10}$, &c. or in the 4th Order by multiplying by $-\frac{1}{2}$, $-\frac{3}{4}$, $-\frac{5}{6}$, $-\frac{7}{8}$, $-\frac{9}{10}$, &c. But the co-efficients of the denominators 1, 3, 15, 105, &c. ariſe by multiplying the numbers 1, 3, 5, 7, 9, &c. gradually into each other.

75. But in the 2d Table, the Series of the 1ſt, 2d, 3d, 4th, 9th, and 10th Orders are produced *in infinitum* by diviſion alone. Thus having $\frac{dz^{4\eta-1}}{e+fz^{\eta}} = y$, in the 1ſt Order, if you perform the diviſion to a convenient period, there will ariſe $\frac{d}{f}z^{3\eta-1} - \frac{de}{ff}z^{2\eta-1} + \frac{de^2}{f^3}z^{\eta-1} - \frac{\frac{de^3}{f^3}z^{\eta-1}}{e+fz^{\eta}} = y$. The firſt three terms belong to the 1ſt Order of Table 1, and the fourth term belongs to the 1ſt Species of this Order. Whence it appears, that the Area is $\frac{d}{3\eta f}z^{3\eta} - \frac{de}{2\eta ff}z^{2\eta} + \frac{de^2}{\eta f^3}z^{\eta} - \frac{e^3}{\eta f^3}s$; putting s for the Area of the Conic Section, whoſe Abſciſs is $x = z^{\eta}$, and Ordinate $v = \frac{d}{e+fx}$.

76. But the Series of the 5th and 6th Orders may be infinitely continued, by the help of the two Theorems in the 5th Order of Table 1. by a due addition or ſubtraction: As alſo the 7th and 8th Series, by means of the Theorems in the 6th Order of Table 1. and the Series of the 11th, by the Theorem in the 10th Order of Table 1.

For inſtance, if the Series of the 3d Order of Table 2. be to be farther continued, ſuppoſe $\theta = -4n$, and the 1ſt Theorem of the 5th Order of Table 1. will become $-8nez^{-4n-1} - 5nfz^{-3n-1}$ into $\frac{1}{3}\sqrt{e+fz^{n}} = y$. $\frac{R^3}{z^{4n}} = t$. But according to the 4th Theorem of this Series to be produced, writing $-\frac{5nf}{2}$ for d, it is $-\frac{5n}{2}fz^{-3n-1}\sqrt{e+fz^{n}} = y$, $\frac{1}{z^{n}} = x$, $\sqrt{fx+exx} = v$, and $\frac{10fv^3-15f^2s}{12e} = t$. So that ſubtracting the former values of y and t, there will remain $4nez^{-4n-1}\sqrt{e+fz^{n}} = y$, $\frac{10fv^3-15ffs}{12e} - \frac{R^3}{z^{4n}} = t$. Theſe being multiplied by $\frac{d}{4ne}$; and, (if you pleaſe) for $\frac{R^3}{z^{4n}}$ writing xv^3, there will ariſe a 5th Theorem of the Series to be produced, $\frac{d}{z^{4n+1}}\sqrt{e+fz^{n}} = y$, $\frac{1}{z^{n}} = x$, $\sqrt{fx+exx} = v$, and $\frac{10dfv^3-15df^2s}{48ne^2} - \frac{dxv^3}{4ne} = t$.

77. IV. Some of theſe Orders may alſo be otherwiſe derived from others. As in the 2d Table, the 5th, 6th, 7th, and 11th, from the 8th; and the 9th from the 10th: So that I might have omitted them, but that they may be of ſome uſe, tho' not altogether neceſſary. Yet I have omitted ſome Orders, which I might have derived from the 1ſt, and 2d, as alſo from the 9th and 10th, becauſe they were affected by Denominators that were more complicate, and therefore can hardly be of any uſe.

78. V. If the defining Equation of any Curve is compounded of ſeveral Equations of different Orders, or of different Species of the ſame Order, its Area muſt be compounded of the correſponding Areas; taking care however, that they may be rightly connected with their proper Signs. For we muſt not always add or ſubtract at the ſame time Ordinates to or from Ordinates, or correſponding Areas to or from correſponding Areas; but ſometimes the ſum of theſe, and the difference of thoſe, is to be taken for a new Ordinate, or to conſtitute a correſponding Area. And this muſt be done, when the conſtituent Areas are poſited on the contrary ſide of the Ordinate. But that the cautious Geometrician may the more readily avoid this in-

inconveniency, I have prefix'd their proper Signs to the ſeveral Values of the Areas, tho' ſometimes negative, as is done in the 5th and 7th Order of Table 2.

79. VI. It is farther to be obſerved, about the Signs of the Areas, that $+s$ denotes, either that the Area of the Conic Section, adjoining to the Abſciſs, is to be added to the other quantities in the value of t; (ſee the 1ſt Example following;) or that the Area on the other ſide of the Ordinate is to be ſubtracted. And on the contrary, $-s$ denotes ambiguouſly, either that the Area adjacent to the Abſciſs is to be ſubtracted, or that the Area on the other ſide of the Ordinate is to be added, as it may ſeem convenient. Alſo the Value of t, if it comes out affirmative, denotes the Area of the Curve propoſed adjoining to its Abſciſs: And contrariwiſe, if it be negative, it repreſents the Area on the other ſide of the Ordinate.

80. VII. But that this Area may be more certainly defined, we muſt enquire after its Limits. And as to its Limit at the Abſciſs, at the Ordinate, and at the Perimeter of the Curve, there can be no uncertainty: But its initial Limit, or the beginning from whence its deſcription commences, may obtain various poſitions. In the following Examples it is either at the beginning of the Abſciſs, or at an infinite diſtance, or in the concourſe of the Curve with its Abſciſs. But it may be placed elſewhere. And wherever it is, it may be found, by ſeeking that length of the Abſciſs, at which the value of t becomes nothing, and there erecting an Ordinate. For the Ordinate ſo raiſed will be the Limit required.

81. VIII. If any part of the Area is poſited below the Abſciſs, t will denote the difference of that, and of the part above the Abſciſs.

82. IX. Whenever the dimenſions of the terms in the values of x, v, and t, ſhall aſcend too high, or deſcend too low, they may be reduced to a juſt degree, by dividing or multiplying ſo often by any given quantity, which may be ſuppos'd to perform the office of Unity, as often as thoſe dimenſions ſhall be either too high or too low.

83. X. Beſides the foregoing Catalogues, or Tables, we might alſo conſtruct Tables of Curves related to other Curves, which may be the moſt ſimple in their kind; as to $\sqrt{a+fx^3}=v$, or to $x\sqrt{e+fx^3}=v$, or to $\sqrt{e+fx^4}=v$, &c. So that we might at all times derive the Area of any propoſed Curve from the ſimpleſt original, and know to what Curves it ſtands related. But now let us illuſtrate by Examples, what has been already delivered.

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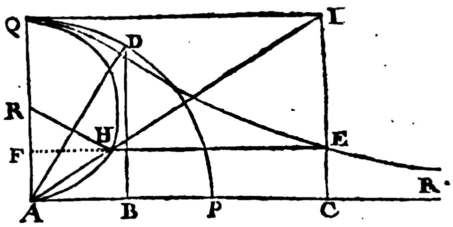


84. EXAMPLE I. Let QER be a Conchoidal of ſuch a kind, that the Semicircle QHA being deſcribed, and AC being erected perpendicular to the Diameter AQ; if the Parallelogram QACI be compleated, the Diagonal AI be drawn, meeting the Semicircle in H, and from H the perpendicular HE be let fall to IC; then the Point E will deſcribe a Curve, whoſe Area ACEQ is ſought.

85. Therefore make AQ $= a$, AC $= z$, CE $= y$; and becauſe of the continual Proportionals AI, AQ, AH, EC, 'twill be EC or $y = \frac{a^3}{a^2+z^2}$.

86. Now that this may acquire the Form of the Equations in the Tables, make $n = 2$, and for z^2 in the denominator write z^n, and $a^3z^{\frac{1}{2}n-1}$ for a^3 or a^3z^{1-1} in the numerator, and there will ariſe $y = \frac{a^3z^{\frac{1}{2}n-1}}{a^2+z^n}$, an Equation of the 1ſt Species of the 2d Order of Table 2. and the Terms being compared, it will be $d = a^3$, $e = a^2$, and $f = 1$; ſo that $\sqrt{\frac{a^3}{a^2+z^2}} = x$, $\sqrt{a^3 - a^2x^2} = v$, and $xv - 2s = t$.

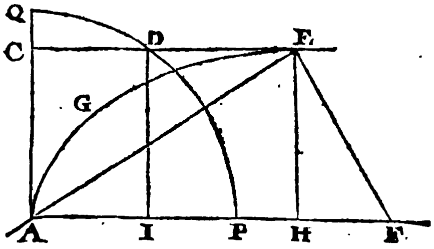
87. Now that the values found of x and v may be reduced to a juſt number of dimenſions, chooſe any given quantity, as a, by which, as unity, a^3 may be multiplied once in the value of x, and in the value of v, a^3 may be divided once, and a^2x^2 twice. And by this means you will obtain $\sqrt{\frac{a^4}{a^2+z^2}} = x$, $\sqrt{a^2 - x^2} = v$, and $xv - 2s, = t$: of which the conſtruction is thus.

88. Center A, and Radius AQ, deſcribe the Quadrantal Arch QDP; in AC take AB $=$ AH; raiſe the perpendicular BD meeting that Arch in D, and draw AD. Then the double of the Sector ADP will be equal to the Area ſought ACEQ. For $\sqrt{\frac{a^4}{a^2+z^2}} =$ ($\sqrt{ADq - ABq} =$) BD, or v; and $xv - 2s = 2\triangle$ ADB-2ABDQ, or $= 2\triangle$ ADB $+$ 2BDP, that is, either $= -2$QAD, or $= 2$DAP: Of which values the affirmative 2DAP belongs to the Area ACEQ on this ſide EC, and the negative -2QAD belongs to the Area RECR extended *ad infinitum* beyond EC.

89. The ſolutions of Problems thus found may ſometimes be made more elegant. Thus in the preſent caſe, drawing RH the ſe-

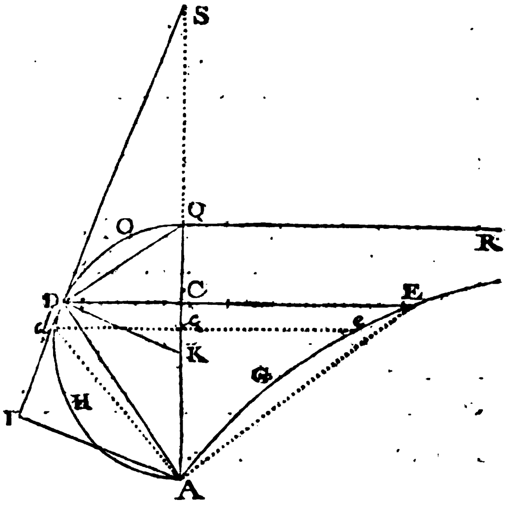
midiameter of the Circle QHA, becauſe of equal Arches QH and DP, the Sector QRH is half the Sector DAP, and therefore a fourth part of the Surface ACEQ.

90. EXAMPLE II. Let AGE be a Curve, which is deſcribed by the Angular point E of the *Norma* AEF, whilſt one of the Legs AE, being interminate, paſſes continually through the given point A, and the other CE, of a given length, ſlides upon the right Line AF given in poſition. Let fall EH perpendicular to AF, and compleat the Parallelogram AHEC; and calling AC $= z$, CE $= y$, and EF $= a$, becauſe of HF, HE, HA continual Proportionals, it will be HA or $y = \frac{z^2}{\sqrt{a^2 - z^2}}$.

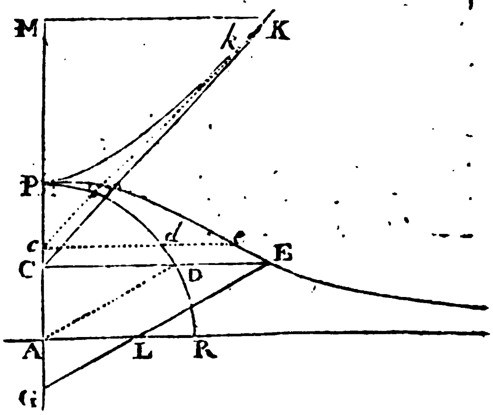


91. Now that the Area AGEC may be known, ſuppoſe $z^2 = z^\eta$, or $2 = \eta$, and thence it will be $\frac{z^{\frac{3}{2}\eta - 1}}{\sqrt{a^2 - z^\eta}} = y$. Here ſince z in the numerator is of a fracted dimenſion, depreſs the value of y by dividing by $z^{\frac{1}{2}\eta}$, and it will be $\frac{z^{\eta - 1}}{\sqrt{a^2 z^{-\eta} - 1}} = y$, an Equation of the 2d Species of the 7th Order of Table 2. And the terms being compared, it is $d = 1$, $e = -1$, and $f = a^2$. So that $z^2 = \left(\frac{1}{z^{-\eta}} =\right) x^2$, $\sqrt{a^2 - x^2} = v$, and $s - xv = t$. Therefore ſince x and z are equal, and ſince $\sqrt{a^2 - x^2} = v$ is an Equation to a Circle, whoſe Diameter is a: with the Center A, and diſtance a or EF, let the Circle PDQ be deſcribed, which CE meets in D, and let the Parallelogram ACDI be compleated; then will AC $= z$, CD $= v$, and the Area ſought AGEC $= s - xv =$ ACDP $-$ ACDI $=$ IDP.

92. EXAMPLE III. Let AGE be the Ciſſoid belonging to the Circle ADQ, deſcribed with the diameter AQ. Let DCE be drawn perpendicular to the diameter, and meeting the Curves in D and E. And naming AC $= z$, CE $= y$, and AQ $= a$; becauſe of CD, CA, CE continual Proportionals, it will be CE or $y = \frac{zz}{\sqrt{az-zz}}$, and dividing by z, 'tis $y = \frac{z}{\sqrt{az^{-1}-1}}$. Therefore $z^{-1} = z^{n}$, or $-1 = n$, and thence $y = \frac{z^{-2n-1}}{\sqrt{az^{n}-1}}$, an Equation of the 3d Species of the 4th Order of Table 2. The Terms therefore being compared, 'tis $d = 1$, $e = -1$, and $f = a$. Therefore $z = \frac{1}{z^{n}} = x$, $\sqrt{ax-xx} = v$, and $3s - 2xv = t$. Wherefore it is AC $= x$, CD $= v$, and thence ACDH $= s$; ſo that 3ACDH $- 4\Delta$ADC $= 3s - 2xv = t =$ Area of the Ciſſoid ACEGA. Or, which is the ſame thing, 3 Segments ADHA $=$ Area ADEGA, or 4 Segments ADHA $=$ Area AHDEGA.



93. EXAMPLE IV. Let PE be the firſt Conchoid of the Ancients, deſcribed from Center G, with the Aſymptote AL, and diſtance LE. Draw its Axis GAP, and let fall the Ordinate EC. Then calling AC $= z$, CE $= y$, GA $= b$, and AP $= c$; becauſe of the Proportionals AC : CE — AL :: GC : CE, it will be CE or $y = \frac{b+z}{z}\sqrt{c^{2}-z^{2}}$.



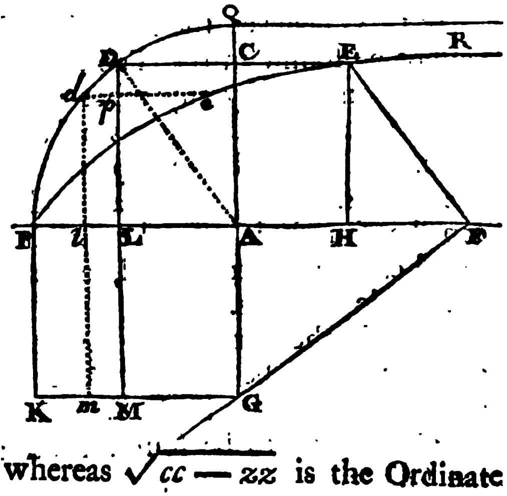
94. Now that its Area PEC may be found from hence, the parts of the Ordinate CE are to be conſider'd ſeparately. And if the Ordinate CE is ſo divided in D, that it is CD $= \sqrt{c^{2}-z^{2}}$,

and DE $= \frac{b}{z}\sqrt{c^2 - z^2}$; CD will be the Ordinate of a Circle deſcribed from Center A, and with the Radius AP. Therefore the part of the Area PDC is known, and there will remain the other part DPED to be found. Therefore ſince DE, the part of the Ordinate by which it is deſcribed, is equivalent to $\frac{b}{z}\sqrt{c^2 - z^2}$; ſuppoſe $2 = \eta$, and it becomes $\frac{b}{z}\sqrt{c^2 - z^\eta} =$ DE, an Equation of the 1ſt Species of the 3d Order of Table 2. The terms therefore being compared, it is $d = b$, $e = c^2$, and $f = -1$; and therefore $\frac{1}{z} = \sqrt{\frac{1}{z^\eta}} = x$, $\sqrt{-1 + c^2x^2} = v$, and $2bc^2s - \frac{bv^3}{x} = t$.

95. Theſe things being found, reduce them to a juſt number of dimenſions, by multiplying the terms that are too depreſs'd, and dividing thoſe that are too high, by ſome given Quantity. If this be done by c, there will ariſe $\frac{c^2}{z} = x$, $\sqrt{-c^2 + x^2} = v$, and $\frac{2bs}{c} - \frac{bv^3}{cx} = t$: The Conſtruction of which is in this manner.

96. With the Center A, principal Vertex P, and Parameter 2AP, deſcribe the Hyperbola PK. Then from the point C draw the right Line CK, that may touch the Parabola in K: And it will be, as AP to 2AG, ſo is the Area CKPC to the ~~Area re~~quired DPED.

97. EXAMPLE 5. Let the *Norma* GFE ſo revolve about the Pole G, as that its angular point F may continually ſlide upon the right Line AF given in poſition; then conceive the Curve PE to be deſcribed by any Point E in the other Leg EF. Now that the Area of this Curve may be found, let fall GA and EH perpendicular to the right Line AF, and compleating the Parallelogram AHEC, call AC $= z$, CE $= y$, AG $= b$, and EF $= c$; and becauſe of the Proportionals HF : EH :: AG : AF, we ſhall have AF $= \frac{bz}{\sqrt{cc - zz}}$. Therefore CE or y $= \frac{bz}{\sqrt{c^2 - z^2}} - \sqrt{c^2 - z^2}$. But whereas $\sqrt{cc - zz}$ is the Ordinate of a Circle deſcribed with the Semidiameter c; about the Center A



let ſuch a Circle PDQ be deſcribed, which CE produced meets in D; then it will be DE $= \frac{bz}{\sqrt{c^2 - z^2}}$: By the help of which Equation there remains the Area PDEP or DERQ to be determin'd. Suppoſe therefore $\eta = 2$, and $\theta = b$, and it will be DE $= \frac{bz^{\eta - 1}}{\sqrt{cc - z^{\eta}}}$, an Equation of the 1ſt Species of the 4th Order of Table 1. And the Terms being compared, it will be $b = d$, $cc = e$, and $-1 = f$; ſo that $-b\sqrt{cc - zz} = -b\text{R} = t$.

98. Now as the value of t is negative, and therefore the Area repreſented by t lies beyond the Line DE; that its initial Limit may be found, ſeek for that length of z, at which t becomes nothing, and you will find it to be c. Therefore continue AC to Q, that it may be AQ $= c$, and erect the Ordinate QR; and DQRED will be the Area whoſe value now found is $-b\sqrt{cc - zz}$.

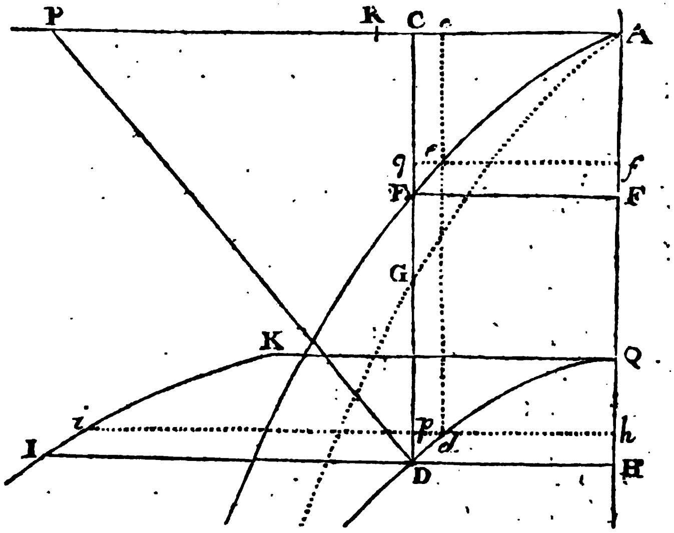
99. If you ſhould deſire to know the quantity of the Area PDE, poſited at the Abſciſs AC, and co-extended with it, without knowing the Limit QR, you may thus determine it.

100. From the Value which t obtains at the length of the Abſciſs AC, ſubtract its value at the beginning of the Abſciſs; that is, from $-b\sqrt{cc - zz}$ ſubtract $-bc$, and there will ariſe the deſired quantity $bc - b\sqrt{cc - zz}$. Therefore compleat the Parallelogram PAGK, and let fall DM perpendicular to AP, which meets GK in M; and the Parallelogram PKML will be equal to the Area PDE.

101. Whenever the Equation defining the nature of the Curve cannot be found in the Tables, nor can be reduced to ſimpler terms by diviſion, nor by any other means; it muſt be transform'd into other Equations of Curves related to it, in the manner ſhewn in Prob. 8. till at laſt one is produced, whoſe Area may be known by the Tables. And when all endeavours are uſed, and yet no ſuch can be found, it may be certainly concluded, that the Curve propoſed cannot be compared, either with rectilinear Figures, or with the Conic Sections.

102. In the ſame manner when mechanical Curves are concern'd, they muſt firſt be transform'd into equal Geometrical Figures, as is ſhewn in the ſame Prob. 8. and then the Areas of ſuch Geometrical Curves are to be found from the Tables. Of this matter take the following Example.

103. EXAMPLE 6. Let it be propoſed to determine the Area of the Figure of the Arches of any Conic Section, when they are made Ordinates on their Right Sines. As let A be the Center of the Conic Section, AQ and AR the Semiaxes, CD the Ordinate to the Axis AR, and PD a Perpendicular at the point D. Alſo let AE be the ſaid mechanical Curve meeting CD in E; and from its nature before defined, CE will be equal to the Arch QD. Therefore the Area AEC is ſought, or compleating the parallelogram ACEF, the exceſs AEF is required. To which purpoſe let a be the Latus rectum of the Conic Section, and b its Latus tranſverſum, or 2AQ. Alſo let AC $= z$, and CD $= y$; then it will be $\sqrt{\frac{1}{4}bb + \frac{b}{a}zz} = y$, an Equation to a Conic Section, as is known. Alſo PC $= \frac{b}{a}z$, and thence PD $= \sqrt{\frac{1}{4}bb + \frac{bb+ab}{aa}zz}$.



104. Now ſince the fluxion of the Arch QD is to the fluxion of the Abſciſs AC, as PD to CD; if the fluxion of the Abſciſs be ſuppos'd 1, the Fluxion of the Arch QD, or of the Ordinate CE, will be $\sqrt{\dfrac{\frac{1}{4}bb + \frac{bb+ab}{aa}zz}{\frac{1}{4}bb + \frac{b}{a}zz}}$. Draw this into FE, or z, and there will ariſe $z\sqrt{\dfrac{\frac{1}{4}bb + \frac{bb+ab}{aa}zz}{\frac{1}{4}bb + \frac{b}{a}zz}}$ for the fluxion of the Area AEF. If therefore in the Ordinate CD you take CG $= z\sqrt{\dfrac{\frac{1}{4}bb + \frac{bb+ab}{aa}zz}{\frac{1}{4}bb + \frac{b}{a}zz}}$, the Area AGC, which is deſcribed by CG moving upon AC, will be equal to the Area AEF, and the Curve

AG will be a Geometrical Curve. Therefore the Area AGC is ſought. To this purpoſe let z^{n} be ſubſtituted for z^{η} in the laſt Equation, and it becomes $z^{n-1}\sqrt{\frac{\frac{1}{4}bb+\frac{bb+ab}{aa}z^{n}}{\frac{1}{4}bb+\frac{b}{a}z^{n}}}=\text{CG}$; an Equation of the 2d Species of the 11th Order of Table 2. And from a compariſon of terms it is $d=1$, $e=\frac{1}{4}bb=g$, $f=\frac{bb+ab}{aa}$, and $b=\frac{b}{a}$; ſo that $\sqrt{\frac{1}{4}bb+\frac{b}{a}zz}=x$, $\sqrt{-\frac{b^{3}}{4a}+\frac{a+b}{a}xx}=u$, and $\frac{a}{b}s=t$. That is, CD $=x$, DP $=v$, and $\frac{a}{b}s=t$. And this is the Conſtruction of what is now found.

105. At Q erect QK perpendicular and equal to QA, and thro' the point D draw HI parallel to it, but equal to DP. And the Line KI, at which HI is terminated, will be a Conic Section, and the comprehended Area HIKQ will be to the Area ſought AEF, as b to a, or as PC to AC.

106. Here obſerve, that if you change the ſign of b, the Conic Section, to whoſe Arch the right Line CE is equal, will become an Ellipſis; and beſides, if you make $b=-a$, the Ellipſis becomes a Circle. And in this caſe the line KI becomes a right line parallel to AQ.

107. After the Area of any Curve has been thus found and conſtructed, we ſhould conſider about the demonſtration of the conſtruction; that laying aſide all Algebraical calculation, as much as may be, the Theorem may be adorn'd, and made elegant, ſo as to become fit for publick view. And there is a general method of demonſtrating, which I ſhall endeavour to illuſtrate by the following Examples.

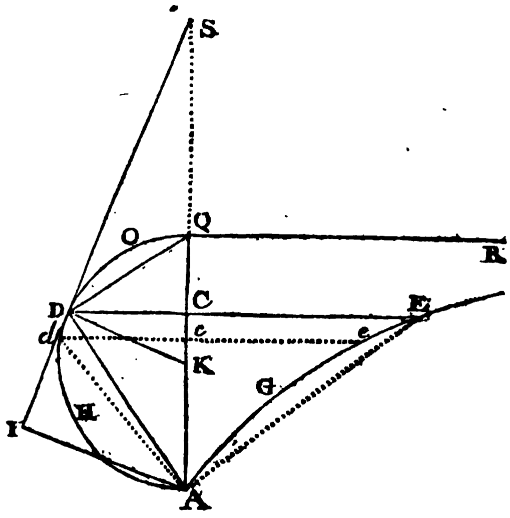
Demonſtration of the Conſtruction in Example 5.

108. In the Arch PQ take a point d indefinitely near to D, (Figure p. 113.) and draw de and dm parallel to DE and DM, meeting DM and AP in p and l. Then will DEed be the moment of the Area PDEP, and LMml will be the moment of the Area LMKP. Draw the ſemidiameter AD, and conceive the indefinitely ſmall arch Dd to be as it were a right line, and the triangles Dpd and ALD will be like, and therefore Dp : pd :: AL : LD. But it is HF : EH :: AG : AF; that is, AL : LD :: ML : DE; and therefore Dp : pd :: ML : DE. Wherefore Dp × DE $=$ pd × ML.

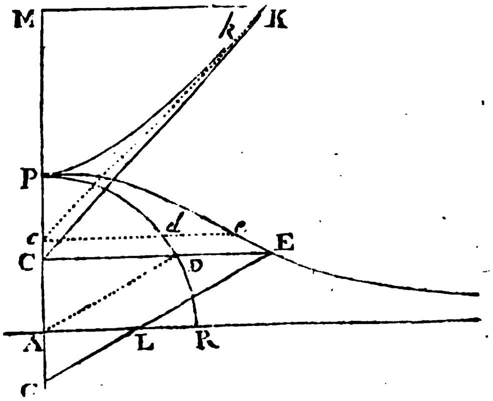
That is, the moment DE*ed* is equal to the moment LM*ml*. And ſince this is demonſtrated indeterminately of any contemporaneous moments whatever, it is plain, that all the moments of the Area PDEP are equal to all the contemporaneous moments of the Area PLMK, and therefore the whole Areas compoſed of thoſe moments are equal to each other. Q. E. D.

Demonſtration of the Conſtruction in Example 3.

109. Let DE*ed* be the momentum of the ſuperficies AHDE, and A*d*DA be the contemporary moment of the Segment ADH. Draw the ſemidiameter DK, and let *de* meet AK in *c*; and it is C*c* : D*d* :: CD : DK. Beſides it is DC : QA (2DK) :: AC : DE. And therefore C*c* : 2D*d* :: DC : 2DK :: AC : DE, and C*c* × DE = 2D*d* × AC. Now to the moment of the periphery D*d* produced, that is, to the tangent of the Circle, let fall the perpendicular AI, and AI will be equal to AC. So that 2D*d* × AC = 2D*d* × AI = 4 Triangles AD*d*. So that 4 Triangles AD*d* = C*c* × DE = moment DE*ed*. Therefore every moment of the ſpace AHDE is quadruple of the contemporary moment of the Segment ADH, and therefore that whole ſpace is quadruple of the whole Segment. Q. E. D.



Demonſtration of the Conſtruction in Example 4.



110. Draw *ce* parallel to CE, and at an indefinitely ſmall diſtance from it, and the tangent of the Hyperbola *ck*, and let fall KM perpendicular to AP. Now from the nature of the Hyperbola it will be AC : AP ∷ AP : AM, and therefore AG*q* : GL*q* ∷ AC*q* : LE*q* (or AP*q*) ∷ AP*q* : AM*q*; and *diviſim*, AG*q* : AL*q* (DE*q*) ∷ AP*q* : AM*q* — AP*q* (MK*q*); And *inverſè*, AG : AP ∷ DE : MK. But the little Area DE*ed* is to the Triangle CK*c*, as the altitude DE is to half the altitude KM; that is, as AG to $\frac{1}{2}$AP. Wherefore all the moments of the Space PDE are to all the contemporaneous moments of the Space PKC, as AG to $\frac{1}{2}$AP. And therefore thoſe whole Spaces are in the ſame ratio. Q. E. D.

Demonſtration of the Conſtruction in Example 6.

111. Draw *cd* parallel and infinitely near to CD, (Fig. in p. 115.) meeting the Curve AE in *e*, and draw *bi* and *fe* meeting DC in *p* and *q*. Then by the Hypotheſis D*d* = E*q*, and from the ſimilitude of the Triangles D*dp* and DCP, it will be D*p* : (D*d*) E*q* ∷ CP : (PD) HI, ſo that D*p* × HI = E*q* × CP; and thence D*p* × HI (the moment HI*ib*) : E*q* × AC (the moment EF*fe*) ∷ E*q* × CP : E*q* × AC ∷ CP : AC. Wherefore ſince PC and AC are in the given ratio of the latus tranſverſum to the latus rectum of the Conic Section QD, and ſince the moments HI*ib* and EF*fe* of the Areas HIKQ and AEF are in that ratio, the Areas themſelves will be in the ſame ratio. Q. E. D.

112. In this kind of demonſtrations it is to be obſerved, that I aſſume ſuch quantities for equal, whoſe ratio is that of equality: And that is to be eſteem'd a ratio of equality, which differs leſs from equality than by any unequal ratio that can be aſſign'd. Thus in the laſt demonſtration I ſuppos'd the rectangle E*q* × AC, or FE*qf*, to be equal to the ſpace FE*ef*, becauſe (by reaſon of the difference E*qe* infinitely leſs than them, or nothing in compariſon of them,) they

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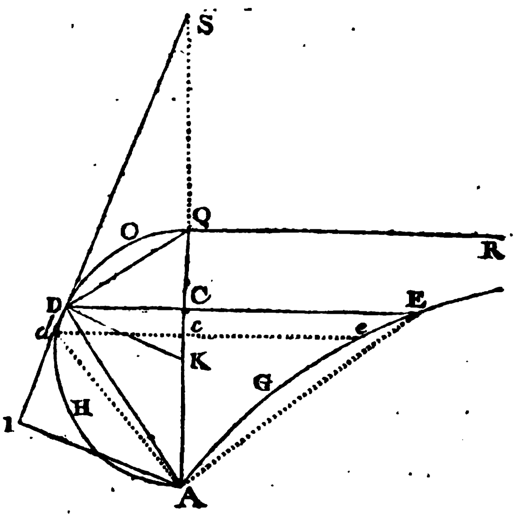
they have not a ratio of inequality. And for the ſame reaſon I made DP × HI = HI*ib*; and ſo in others.

113. I have here made uſe of this method of proving the Areas of Curves to be equal, or to have a given ratio, by the equality, or by the given ratio, of their moments; becauſe it has an affinity to the uſual methods in theſe matters. But that ſeems more natural which depends upon the generation of Superficies, by Motion or Fluxion. Thus if the Conſtruction in Example 2. was to be demonſtrated: From the nature of the Circle, the fluxion of the right line ID (Fig. p. 111.) is to the fluxion of the right line IP, as AI to ID; and it is AI : ID :: ID : CE, from the nature of the Curve AGE; and therefore $CE \times \dot{ID} = ID \times \dot{IP}$. But $CE \times \dot{ID}$ = to the fluxion of the Area PDI. And therefore thoſe Areas, being generated by equal fluxion, muſt be equal. Q. E. D.

114. For the ſake of farther illuſtration, I ſhall add the demonſtration of the Conſtruction, by which the Area of the Ciſſoid is determin'd, in Example 3. Let the lines mark'd with points in the ſcheme be expunged; draw the Chord DQ, and the Aſymptote QR of the Ciſſoid. Then, from the nature of the Circle, it is DQ*q* = AQ × CQ, and thence (by Prob. 1.) 2DQ × Fluxion of DQ = $AQ \times \dot{CQ}$. And therefore $AQ : DQ :: 2\dot{DQ} : \dot{CQ}$. Alſo from the nature of the Ciſſoid it is ED : AD :: AQ : DQ. Therefore $ED : AD :: 2\dot{DQ} : \dot{CQ}$, and $ED \times \dot{CQ} = AD \times 2\dot{DQ}$, or $4 \times \frac{1}{4}AD \times \dot{DQ}$. Now ſince DQ is perpendicular at the end of AD, revolving about



A; and $\frac{1}{4}AD \times \dot{QD}$ = to the fluxion generating the Area ADOQ; its quadruple alſo $ED \times \dot{CQ}$ = fluxion generating the Ciſſoidal Area QREDO. Wherefore that Area QREDO infinitely long, is generated quadruple of the other ADOQ. Q. E. D.

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115. By the foregoing Tables not only the Areas of Curves, but quantities of any other kind, that are generated by an analogous way of flowing, may be derived from their Fluxions, and that by the aſſiſtance of this Theorem: That a quantity of any kind is to an unit of the ſame kind, as the Area of a Curve is to a ſuperficial unity; if ſo be that the fluxion generating that quantity be to an unit of its kind, as the fluxion generating the Area is to an unit of its kind alſo; that is, as the right Line moving perpendicularly upon the Abſciſs (or the Ordinate) by which the Area is deſcribed, to a linear Unit. Wherefore if any fluxion whatever is expounded by ſuch a moving Ordinate, the quantity generated by that fluxion will be expounded by the Area deſcribed by ſuch Ordinate; or if the Fluxion be expounded by the ſame Algebraic terms as the Ordinate, the generated quantity will be expounded by the ſame as the deſcribed Area. Therefore the Equation, which exhibits a Fluxion of any kind, is to be ſought for in the firſt Column of the Tables, and the value of t in the laſt Column will ſhow the generated Quantity.

116. As if $\sqrt{1+\frac{9z}{4a}}$ exhibited a Fluxion of any kind, make it equal to y, and that it may be reduced to the form of the Equations in the Tables, ſubſtitute z^n for z, and it will be $z^{n-1}\sqrt{1+\frac{9}{4a}z^n}=y$, an Equation of the firſt Species of the 3d Order of Table 1. And comparing the terms, it will be $d=1$, $e=1$, $f=\frac{9}{4a}$, and thence $\frac{8a+18z}{27}\sqrt{1+\frac{9z}{4a}}=\frac{2d}{3nf}R^3=t$. Therefore it is the quantity $\frac{8a+18z}{27}\sqrt{1+\frac{9z}{4a}}$ which is generated by the Fluxion $\sqrt{1+\frac{9z}{4a}}$.

117. And thus if $\sqrt{1+\frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$ repreſents a Fluxion, by a due reduction, (or by extracting $z^{\frac{2}{3}}$ out of the radical, and writing z^n for $z^{-\frac{2}{3}}$,) there will be had $\frac{1}{z^{n+1}}\sqrt{z^n+\frac{16}{9a^{\frac{2}{3}}}}=y$, an Equation of the 2d Species of the 5th Order of Table 2. Then comparing the

terms, it is $d = 1$, $e = \frac{16}{9a^{\frac{2}{3}}}$, and $f = 1$. So that $z^{\frac{2}{3}} = \frac{1}{z^{n}} = xx$, $\sqrt{1 + \frac{16xx}{9a^{\frac{2}{3}}}} = v$, and $\frac{4}{3}s = \frac{-2d}{n}s = t$. Which being found, the quantity generated by the fluxion $\sqrt{1 + \frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$ will be known, by making it to be to an Unit of its own kind, as the Area $\frac{4}{3}s$ is to ſuperficial unity; or which comes to the ſame, by ſuppoſing the quantity t no longer to repreſent a Superficies, but a quantity of another kind, which is to an unit of its own kind, as that ſuperficies is to ſuperficial unity.

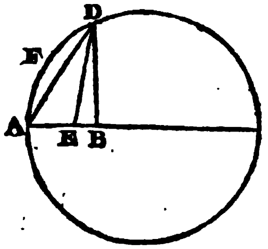
118. Thus ſuppoſing $\sqrt{1 + \frac{16x^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$ to repreſent a linear Fluxion, I imagine t no longer to ſignify a Superficies, but a Line; that Line, for inſtance, which is to a linear unit, as the Area which (according to the Tables) is repreſented by t, is to a ſuperficial unit, or that which is produced by applying that Area to a linear unit. On which account, if that linear unit be made e, the length generated by the foregoing fluxion will be $\frac{3s}{2e}$. And upon this foundation thoſe Tables may be apply'd to the determining the Lengths of Curve-lines, the Contents of their Solids, and any other quantities whatever, as well as the Areas of Curves.

Of Queſtions that are related hereto.

I. *To approximate to the Areas of Curves mechanically.*

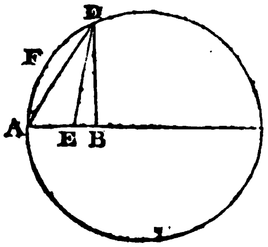
119. The method is this, that the values of two or more right-lined Figures may be ſo compounded together, that they may very nearly conſtitute the value of the Curvilinear Area required.

120. Thus for the Circle AFD which is denoted by the Equation $x - xx = zz$, having found the value of the Area AFDB, *viz.* $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}}$, &c. the values of ſome Rectangles are to be ſought, ſuch is the value $x\sqrt{x - xx}$, or $x^{\frac{3}{2}} - \frac{1}{2}z^{\frac{5}{2}} - \frac{1}{8}x^{\frac{7}{2}} - \frac{1}{16}x^{\frac{9}{2}}$, &c. of the rectangle BD × AB, and $x\sqrt{x}$, or $x^{\frac{3}{2}}$, the value of AD × AB. Then theſe values are to be multiply'd by any different letters, that ſtand for numbers indefinitely, and then



to be added together, and the terms of the ſum are to be compared with the correſponding terms of the value of the Area AFDB, that as far as is poſſible they may become equal. As if thoſe Parallelograms were multiply'd by e and f, the ſum would be $\begin{array}{l} ex^{\frac{3}{2}} \\ +f \end{array} - \frac{1}{2}ex^{\frac{5}{2}} - \frac{1}{8}ex^{\frac{7}{2}}$, &c. the terms of which being compared with theſe terms $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}}$, &c. there ariſes $e + f = \frac{2}{3}$, and $-\frac{1}{2}e = -\frac{1}{5}$, or $e = \frac{2}{5}$, and $f = \frac{2}{3} - e = \frac{4}{15}$. So that $\frac{2}{5}$BD × AB + $\frac{4}{15}$AD × AB = Area AFDB very nearly. For $\frac{2}{5}$BD × AB + $\frac{4}{15}$AD × AB is equivalent to $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{20}x^{\frac{7}{2}} - \frac{1}{40}x^{\frac{9}{2}}$, &c. which being ſubtracted from the Area AFDB, leaves the error only $\frac{1}{70}x^{\frac{7}{2}} + \frac{1}{90}x^{\frac{9}{2}}$, &c.

121. Thus if AB were biſected in E, the value of the rectangle AB × DE will be $x\sqrt{x - \frac{3}{4}xx}$, or $x^{\frac{3}{2}} - \frac{3}{8}x^{\frac{5}{2}} - \frac{9}{128}x^{\frac{7}{2}} - \frac{27}{1024}x^{\frac{9}{2}}$, &c. And this compared with the rectangle AD × AB, gives $\frac{8\text{DE} + 2\text{AD}}{15}$ into AB = Area AFDB, the error being only $\frac{1}{560}x^{\frac{7}{2}} + \frac{1}{5760}x^{\frac{9}{2}}$, &c. which is always leſs than $\frac{1}{1500}$ part of the whole Area, even tho' AFDB were a quadrant of a Circle. But this Theorem may be thus propounded. As 3 to 2, ſo is the rectangle AB into DE, added to a fifth part of the difference between AD and DE, to the Area AFDB, very nearly.



122. And thus by compounding two rectangles AB × ED and AB × BD, or all the three rectangles together, or by taking in ſtill more rectangles, other Rules may be invented, which will be ſo much the more exact, as there are more Rectangles made uſe of. And the ſame is to be underſtood of the Area of the Hyperbola, or of any other Curves. Nay, by one only rectangle the Area may often be very commodiouſly exhibited, as in the foregoing Circle, by taking BE to AB as $\sqrt{10}$ to 5, the rectangle AB × ED will be to the Area AFDB, as 3 to 2, the error being only $\frac{1}{175}x^{\frac{7}{2}} + \frac{1}{1120}x^{\frac{9}{2}}$, &c.

II. *The Area being given, to determine the Abſciſs and Ordinate.*

123. When the Area is expreſs'd by a finite Equation, there can be no difficulty: But when it is expreſs'd by an infinite Series, the affected root is to be extracted, which denotes the Abſciſs. So for

the Hyperbola, defined by the Equation $\frac{ab}{a+x} = \dot{z}$, after we have found $z = bx - \frac{bx^2}{2a} + \frac{bx^3}{3a^2} - \frac{bx^4}{4a^3}$, &c. that from the given Area the Abſciſs x may be known, extract the affected Root, and there will ariſe $x = \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{96a^4b^5}$, &c. And moreover, if the Ordinate $\dot{z}$ were required, divide ab by $a+x$, that is, by $a + \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3}$, &c. and there will ariſe $\dot{z} = b - \frac{z}{a} - \frac{z^2}{2a^2b} - \frac{z^3}{6a^3b^2} - \frac{z^4}{24a^4b^3}$, &c.

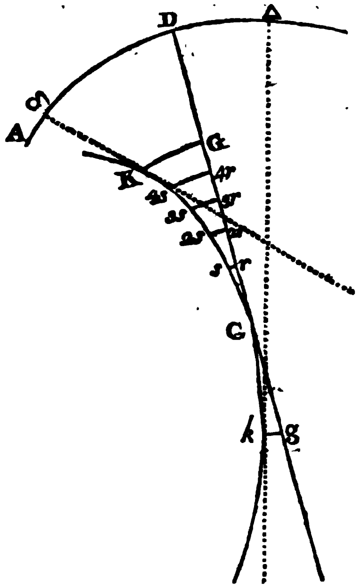
124. Thus as to the Ellipſis which is expreſs'd by the Equation $ax - \frac{a}{c}xx = \dot{z}\dot{z}$, after the Area is found $z = \frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{a^{\frac{1}{2}}x^{\frac{5}{2}}}{5c} - \frac{a^{\frac{1}{2}}x^{\frac{7}{2}}}{28c^2} - \frac{a^{\frac{1}{2}}x^{\frac{9}{2}}}{72c^3}$, &c. write v^3 for $\frac{3z}{2a^{\frac{1}{2}}}$, and t for $x^{\frac{1}{2}}$, and it becomes $v^3 = t^3 - \frac{3t^5}{10c} - \frac{3t^7}{56c^2} - \frac{t^9}{48c^3}$, &c. and extracting the root $t = v + \frac{v^3}{10c} + \frac{81v^5}{1400c^2} + \frac{1171v^7}{252000c^3}$, &c. whoſe ſquare $v^2 + \frac{v^4}{5c} + \frac{22v^6}{175c^2} + \frac{823v^8}{7875c^3}$, &c. is equal to x. And this value being ſubſtituted inſtead of x in the Equation $ax - \frac{a}{c}xx = \dot{z}\dot{z}$, and the root being extracted, there ariſes $\dot{z} = a^{\frac{1}{2}}v - \frac{2a^{\frac{1}{2}}v^3}{5c} - \frac{38a^{\frac{1}{2}}v^5}{175c^2} - \frac{407a^{\frac{1}{2}}v^7}{2250c^3}$, &c. So that from z, the given Area, and thence v or $\sqrt[3]{\frac{3z}{2a^{\frac{1}{2}}}}$, the Abſciſs x will be given, and the Ordinate $\dot{z}$. All which things may be accommodated to the Hyperbola, if only the ſign of the quantity c be changed, wherever it is found of odd dimenſions.

PROB. X.

To find as many Curves as we pleaſe, whoſe Lengths may be expreſs'd by finite Equations.

1. The following poſitions prepare the way for the ſolution of this Problem.

2. I. If the right Line DC, ſtanding perpendicularly upon any Curve AD, be conceived thus to move, all its points G, *g*, *r*, &c. will deſcribe other Curves, which are equidiſtant, and perpendicular to that line: As GK, *gk*, *rs*, &c.



3. II. If that right Line is continued indefinitely each way, its extremities will move contrary ways, and therefore there will be a Point between, which will have no motion, but may therefore be call'd the Center of Motion. This Point will be the ſame as the Center of Curvature, which the Curve AD hath at the point D, as is mention'd before. Let that point be C.

4. III. If we ſuppoſe the line AD not to be circular, but unequably curved, ſuppoſe more curved towards δ, and leſs toward Δ; that Center will continually change its place, approaching nearer to the parts more curved, as in K, and going farther off at the parts leſs curved, as in *k*, and by that means will deſcribe ſome line, as KC*k*.

5. IV. The right Line DC will continually touch the line deſcribed by the Center of Curvature. For if the Point D of this line moves towards δ, its point G, which in the mean time paſſes to K, and is ſituate on the ſame ſide of the Center C, will move the ſame way, by poſition 2. Again, if the ſame point D moves towards Δ, the point *g*, which in the mean time paſſes to *k*, and is ſituate on the contrary ſide of the Center C, will move the contrary way, that is, the ſame way that G moved in the former caſe, while it paſs'd to K. Wherefore K and *k* lie on the ſame ſide of the right Line DC. But as K and *k* are taken indefinitely for any points,

points; it is plain that the whole Curve lies on the ſame ſide of the right line DC, and therefore is not cut, but only touch'd by it.

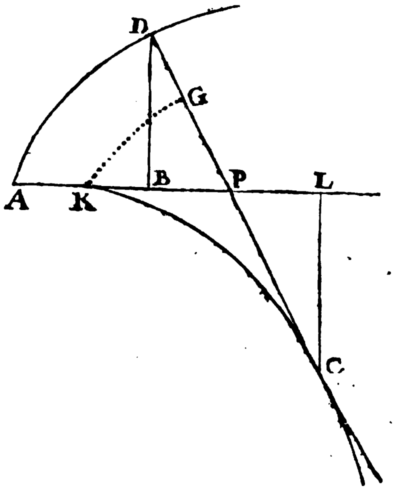
6. Here it is ſuppos'd, that the line δDΔ is continually more curved towards δ, and leſs towards Δ; for if its greateſt or leaſt Curvature is in D, then the right line DC will cut the Curve KC; but yet in an angle that is leſs than any right-lined angle, which is the ſame thing as if it were ſaid to touch it. Nay, the point C in this caſe is the Limit, or Cuſpid, at which the two parts of the Curve, finiſhing in the moſt oblique concourſe, touch each other; and therefore may more juſtly be ſaid to be touch'd, than to be cut, by the right line DC, which divides the Angle of contact.

7. V. The right Line CG is equal to the Curve CK. For conceive all the points r, $2r$, $3r$, $4r$, &c. of that right Line to deſcribe the arches of Curves rs, $2r2s$, $3r3s$, &c. in the mean time that they approach to the Curve CK, by the motion of that right line; and ſince thoſe arches, (by poſition 1.) are perpendicular to the right lines that touch the Curve CK, (by poſition 4.) it follows that they will be alſo perpendicular to that Curve. Wherefore the parts of the line CK, intercepted between thoſe arches, which by reaſon of their infinite ſmallneſs may be conſider'd as right lines, are equal to the intervals of the ſame arches; that is, (by poſition 1.) are equal to ſo many parts of the right line CG. And equals being added to equals, the whole Line CK will be equal to the whole Line CG.

8. The ſame thing would appear by conceiving, that every part of the right Line CG, as it moves along, will apply itſelf ſucceſſively to every part of the Curve CK, and thereby will meaſure them; juſt as the Circumference of a wheel, as it moves forward by revolving upon a Plain, will meaſure the diſtance that the point of Contact continually deſcribes.

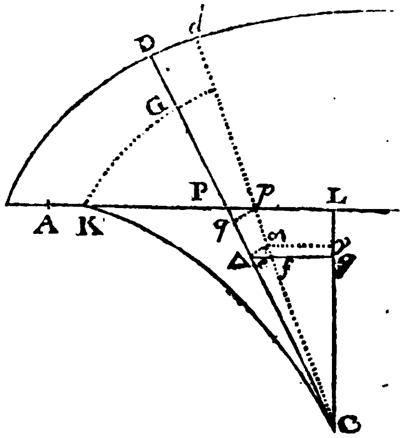
9. And hence it appears, that the Problem may be reſolved, by aſſuming any Curve at pleaſure AδDΔ, and thence by determining the other Curve KCk, in which the Center of Curvature of the aſſumed Curve is always found. Therefore letting fall the perpendiculars DB and CL, to a right Line AB given in poſition, and in AB taking any point A, and calling AB $= x$ and BD $= y$; to define the Curve AD let any relation be aſſumed between x and y, and then by Prob 5. the point C may be found, by which may be determined both the Curve KC, and its Length GC.

10. EXAMPLE. Let $ax = yy$ be the Equation to the Curve, which therefore will be the *Apollonian* Parabola. And, by Prob. 5. will be found $\text{AL} = \frac{1}{2}a + 3x$, $\text{CL} = \frac{4y^3}{aa}$, and $\text{DC} = \frac{a+4x}{a}\sqrt{\frac{1}{4}aa + ax}$. Which being obtain'd, the Curve KC is determin'd by AL and LC, and its Length by DC. For as we are at liberty to aſſume the points K and C any where in the Curve KC, let us ſuppoſe K to be the Center of Curvature of the Parabola at its Vertex; and putting therefore AB and BD, or x and y, to be nothing, it will be $\text{DC} = \frac{1}{2}a$. And this is the Length AK, or DG, which being ſubtracted from the former indefinite value of DC, leaves GC or $\text{KC} = \frac{a+4x}{a}\sqrt{\frac{1}{4}aa + ax} - \frac{1}{2}a$.



11. Now if you deſire to know what Curve this is, and what is its Length, without any relation to the Parabola; call $\text{KL} = z$, and $\text{LC} = v$, and it will be $z = \text{AL} - \frac{1}{2}a = 3x$, or $\frac{1}{3}z = x$, and $\frac{az}{3} = ax = yy$. Therefore $4\sqrt{\frac{z^3}{27a}} = \frac{4y^3}{aa} = \text{CL} = v$, or $\frac{16z^3}{27a} = v^2$; which ſhews the Curve KC to be a Parabola of the ſecond kind. And for its Length there ariſes $\frac{3a+4z}{3a}\sqrt{\frac{1}{4}aa + \frac{1}{3}az} - \frac{1}{2}a$, by writing $\frac{1}{3}z$ for x in the value of CG.

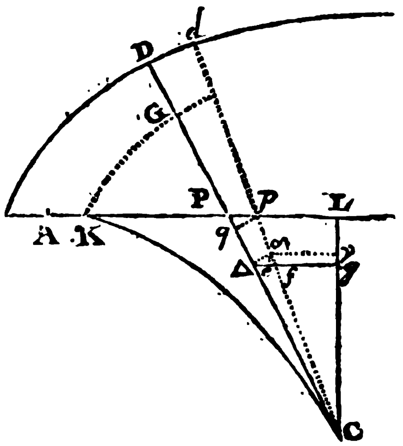
12. The Problem alſo may be reſolved by taking an Equation, which ſhall expreſs the relation between AP and PD, ſuppoſing P to be the interſection of the Abſciſs and Perpendicular. For calling $\text{AP} = x$, and $\text{PD} = y$, conceive CPD to move an infinitely ſmall ſpace, ſuppoſe to the place Cpd, and in CD and Cd taking $\text{C}\Delta$ and $\text{C}\delta$ both of the ſame given length, ſuppoſe $= 1$, and to CL let fall the perpendiculars Δg and $\delta\gamma$, of which Δg, (which call $= z$) may meet Cd in f. Then compleat the Parallelogram $g\gamma\delta e$, and making $\dot{x}$, $\dot{y}$, and $\dot{z}$ the fluxions of the quantities x, y, and z, as before; it



it will be $\Delta e : \Delta f :: \overline{\Delta e}|^2 : \overline{\Delta \delta}|^2 :: \overline{Cg}|^2 : \overline{C\Delta}|^2 :: \frac{\overline{Cg}|^2}{C\Delta} : C\Delta$. And $\Delta f : Pp :: C\Delta : CP$. Then *ex æquo*, $\Delta e : Pp :: \frac{\overline{Cg}|^2}{C\Delta} : CP$. But Pp is the moment of the Abſciſs AP, by the acceſſion of which it becomes Ap; and Δe is the contemporaneous moment of the perpendicular Δg, by the decreaſe of which it becomes $\delta\gamma$. Therefore Δe and Pp are as the fluxions of the lines Δg (z) and AP (x), that is, as $\dot z$ and $\dot x$. Wherefore $\dot z : \dot x :: \frac{\overline{Cg}|^2}{C\Delta} : CP$. And ſince it is $\overline{Cg}|^2 = \overline{C\Delta}|^2 - \overline{\Delta g}|^2 = 1 - zz$, and $C\Delta = 1$; it will be $CP = \frac{\dot x - \dot x z^2}{\dot z}$. Moreover ſince we may aſſume any one of the three $\dot x$, $\dot y$, and $\dot z$ for an uniform fluxion, to which the reſt are to be referr'd, if $\dot x$ be that fluxion, and its value is unity, then $CP = \frac{1 - zz}{\dot z}$.

13. Beſides it is $C\Delta\ (1) : \Delta g\ (z) :: CP : PL$; alſo $C\Delta\ (1) : Cg\ (\sqrt{1 - zz}) :: CP : CL$; therefore it is $PL = \frac{z - z^3}{\dot z}$, and $CL = \frac{1 - zz}{\dot z}\sqrt{1 - zz}$. Laſtly, drawing pq parallel to the infinitely ſmall Arch Dd, or perpendicular to DC, Pq will be the momentum of DP, by the acceſſion of which it becomes dp, at the ſame time that AP becomes Ap. Therefore Pp and Pq are as the fluxions of AP (x) and PD (y), that is, as 1 and $\dot y$. Therefore becauſe of ſimilar triangles Ppq and $C\Delta g$, ſince $C\Delta$ and Δg, or 1 and z, are in the ſame ratio, it will be $\dot y = z$. Whence we have this ſolution of the Problem.

14. From the propoſed Equation, which expreſſes the relation between x and y, find the relation of the fluxions $\dot x$ and $\dot y$, (by Prob. 1.) and putting $\dot x = 1$, there will be had the value of $\dot y$, to which z is equal. Then ſubſtituting z for $\dot y$, by the help of the laſt Equation find the relation of the Fluxions $\dot x$, $\dot y$, and $\dot z$, (by Prob. 1.) and again ſubſtituting 1 for $\dot x$, there will be had the value of z. Theſe being found make $\frac{1 - \dot y\dot y}{\dot z} = CP$, $z \times CP = PL$, and $CP \times \sqrt{1 - \dot y\dot y} = CL$; and C will be a Point in the Curve, any part of which KC is equal to the right Line CG, which is the difference of the tangents, drawn perpendicularly to the Curve Dd from the points C and K.



15. Ex. Let $ax = yy$ be the Equation which expreſſes the relation between AP and PD; and (by Prob. 1.) it will be firſt $a\dot{x} = 2y\dot{y}$, or $a = 2yz$. Then $2\dot{y}z + 2y\dot{z} = 0$, or $\frac{-zz}{y} = \dot{z}$. Thence it is CP $= \frac{1-\dot{y}\dot{y}}{\dot{z}} = y - \frac{4y^3}{aa}$, PL $= z \times$ CP $= \frac{1}{2}a - \frac{2yy}{a}$, and CL $= \frac{aa-4yy}{2aa}\sqrt{4yy-aa}$. And from CP and PL taking away y and x, there remains CD $= -\frac{4y^3}{aa}$, and AL $= \frac{1}{2}a - \frac{3yy}{a}$. Now I take away y and x, becauſe when CP and PL have affirmative values, they fall on the ſide of the point P towards D and A, and they ought to be diminiſhed, by taking away the affirmative quantities PD and AP. But when they have negative values, they will fall on the contrary ſide of the point P, and then they muſt be encreaſed, which is alſo done by taking away the affirmative quantities PD and AP.

16. Now to know the Length of the Curve, in which the point C is found, between any two of its points K and C; we muſt ſeek the length of the Tangent at the point K, and ſubtract it from CD. As if K were the point, at which the Tangent is terminated, when CΔ and Δg, or 1 and z, are made equal, which therefore is ſituate in the Abſciſs itſelf AP; write 1 for z in the Equation $a = 2yz$, whence $a = 2y$. Therefore for y write $\frac{1}{2}a$ in the value of CD, that is in $-\frac{4y^3}{aa}$, and it comes out $-\frac{1}{2}a$. And this is the length of the Tangent at the point K, or of DG; the difference between which and the foregoing indefinite value of CD, is $\frac{4y^3}{aa} - \frac{1}{2}a$, that is GC, to which the part of the Curve KC is equal.

17. Now that it may appear what Curve this is, from AL (having firſt changed its ſign, that it may become affirmative,) take AK, which will be $\frac{1}{4}a$, and there will remain KL $= \frac{3yy}{a} - \frac{1}{4}a$, which call t, and in the value of the line CL, which call v, write $\frac{4at}{3}$ for $4yy - aa$, and there will ariſe $\frac{2t}{3a}\sqrt{\frac{4}{3}at} = v$; or $\frac{16t^3}{27a} = vv$, which is an Equation to a Parabola of the ſecond kind, as was found before.

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18. When the relation between t and v cannot conveniently be reduced to an Equation, it may be ſufficient only to find the lengths PC and PL. As if for the relation between AP and PD the Equation $3a^2x+3a^2y-y^3=0$ were aſſumed; from hence (by Prob. 1.) firſt there ariſes $a^2+a^2z-y^2z=0$, then $aa\dot{z}-2y\dot{y}z-y^2\dot{z}=0$, and therefore it is $z=\frac{aa}{yy-aa}$, and $\dot{z}=\frac{2y\dot{y}z}{aa-yy}$. Whence are given $\mathrm{PC}=\frac{1-zz}{\dot{z}}$, and $\mathrm{PL}=z\times\mathrm{PC}$, by which the point C is determined, which is in the Curve. And the length of the Curve, between two ſuch points, will be known by the difference of the two correſponding Tangents, DC or $\mathrm{PC}-y$.

19. For Example, if we make $a=1$, and in order to determine ſome point C of the Curve, we take $y=2$; then AP or x becomes $\frac{y^3-3a^2y}{3aa}=\frac{2}{3}$, $z=\frac{1}{3}$, $\dot{z}=-\frac{4}{9}$, $\mathrm{PC}=-2$, and $\mathrm{PL}=-\frac{2}{3}$. Then to determine another point, if we take $y=3$, it will be $\mathrm{AP}=6$, $z=\frac{1}{8}$, $\dot{z}=-\frac{3}{256}$, $\mathrm{PC}=-84$, and $\mathrm{PL}=-10\frac{1}{2}$. Which being had, if y be taken from PC, there will remain -4 in the firſt caſe, and -87 in the ſecond, for the lengths DC; the difference of which 83 is the length of the Curve, between the two points found C and c.

20. Theſe are to be thus underſtood, when the Curve is continued between the two points C and c, or between K and C, without that Term or Limit, which we call'd its Cuſpid. For when one or more ſuch terms come between thoſe points, (which terms are found by the determination of the greateſt or leaſt PC or DC,) the lengths of each of the parts of the Curve, between them and the points C or K, muſt be ſeparately found, and then added together.

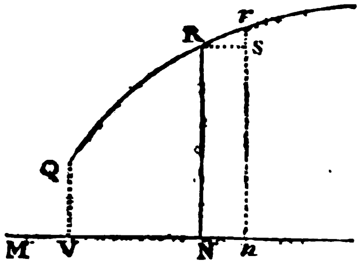
PROB. XI.

To find as many Curves as you pleaſe, whoſe Lengths may be compared with the Length of any Curve propoſed, or with its Area applied to a given Line, by the help of finite Equations.

1. It is performed by involving the Length, or the Area of the propoſed Curve, in the Equation which is aſſumed in the foregoing Problem, to determine the relation between AP and PD (Figure Art. 12. pag. 126.) But that z and $\dot{z}$ may be thence derived, (by

Prob. 1.) the fluxion of the Length, or of the Area, muſt be firſt diſcover'd.

2. The fluxion of the Length is determin'd by putting it equal to the ſquare-root of the ſum of the ſquares of the fluxion of the Abſciſs and of the Ordinate. For let RN be the perpendicular Ordinate, moving upon the Abſciſs MN, and let QR be the propoſed Curve, at which RN is terminated. Then calling MN $=s$, NR $=t$, and QR $=v$, and their Fluxions $\dot{s}$, $\dot{t}$, and $\dot{v}$ reſpectively; conceive the Line NR to move into the place nr infinitely near the former, and letting fall Rs perpendicular to nr, then Rs, sr, and Rr will be the contemporaneous moments of the lines MN, NR, and QR; by the acceſſion of which they become Mn, nr, and Qr. And as theſe are to each other as the fluxions of the ſame lines, and becauſe of the right Angle Rsr, it will be $\sqrt{\text{R}s^2+sr^2}=\text{R}r$, or $\sqrt{\dot{s}^2+\dot{t}^2}=\dot{v}$.



3. But to determine the fluxions $\dot{s}$ and $\dot{t}$ there are two Equations requir'd; one of which is to define the relation between MN and NR, or s and t, from whence the relation between the fluxions $\dot{s}$ and $\dot{t}$ is to be derived; and another which may define the relation between MN or NR in the given Figure, and of AP or x in that required, from whence the relation of the fluxion $\dot{s}$ or $\dot{t}$ to the fluxion $\dot{x}$ or 1 may be diſcover'd.

4. Then $\dot{v}$ being found, the fluxions $\dot{y}$ and $\dot{z}$ are to be ſought by a third aſſumed Equation, by which the length PD or y may be defined. Then we are to take PC $=\frac{1-\dot{y}\dot{y}}{\dot{z}}$, PL $=\dot{y}\times$ PC, and DC $=$ PC $-y$, as in the foregoing Problem.

5. Ex. 1. Let $as-ss=tt$ be an Equation to the given Curve QR, which will be a Circle; $xx=as$ the relation between the lines AP and MN, and $\frac{2}{3}v=y$, the relation between the length of the Curve given QR, and the right Line PD. By the firſt it will be $a\dot{s}-2s\dot{s}=2t\dot{t}$, or $\frac{a-2s}{2t}\dot{s}=\dot{t}$. And thence $\frac{a\dot{s}}{2t}=\sqrt{\dot{s}^2+\dot{t}^2}=\dot{v}$. By the ſecond it is $2x=a\dot{s}$, and therefore $\frac{x}{t}=\dot{v}$. And by the third $\frac{2}{3}\dot{v}=\dot{y}$, that is, $\frac{2x}{3t}=z$, and hence $\frac{2}{3t}-\frac{2x\dot{t}}{3tt}=\dot{z}$. Which

being found, you muſt take $PC = \frac{1 - \dot{y}\dot{y}}{\dot{z}}$, $PL = \dot{y} \times PC$, and $DC = PC - y$, or $PC - \frac{1}{2}QR$. Where it appears, that the length of the given Curve QR cannot be found, but at the ſame time the length of the right Line DC muſt be known, and from thence the length of the Curve, in which the point C is found; and ſo on the contrary.

6. Ex. 2. The Equation $as - ss = tt$ remaining, make $x = s$, and $vv - 4ax = 4ay$. And by the firſt there will be found $-\frac{a\dot{s}}{2t} = \dot{v}$, as above. But by the ſecond $1 = \dot{s}$, and therefore $\frac{a}{2t} = \dot{v}$. And by the third $2v\dot{v} - 4a = 4a\dot{y}$, or (eliminating $\dot{v}$) $\frac{v}{4t} - 1 = z$. Then from hence $\frac{\dot{v}}{4t} - \frac{v\dot{t}}{4tt} = \dot{z}$.

7. Ex. 3. Let there be ſuppos'd three Equations, $aa = st$, $a + 3s = x$, and $x + v = y$. Then by the firſt, which denotes an Hyperbola, it is $0 = \dot{s}t + \dot{t}s$, or $-\frac{\dot{s}t}{s} = \dot{t}$, and therefore $\frac{\dot{s}}{s}\sqrt{ss + tt} = \sqrt{\dot{s}\dot{s} + \dot{t}\dot{t}} = \dot{v}$. By the ſecond it is $3\dot{s} = 1$, and therefore $\frac{1}{3s}\sqrt{ss + tt} = \dot{v}$. And by the third it is $1 + \dot{v} = \dot{y}$, or $1 + \frac{1}{3s}\sqrt{ss + tt} = z$; then it is from hence $\dot{w} = \dot{z}$, that is, putting $\dot{w}$ for the Fluxion of the radical $\frac{1}{3s}\sqrt{ss + tt}$, which if it be made equal to w, or $\frac{1}{9} + \frac{tt}{9s} = ww$, there will ariſe from thence $\frac{2t\dot{t}}{9ss} - \frac{2tt\dot{s}}{9s^3} = 2w\dot{w}$. And firſt ſubſtituting $-\frac{\dot{s}t}{s}$ for $\dot{t}$, then $\frac{1}{3}$ for $\dot{s}$, and dividing by $2w$, there will ariſe $\frac{-2tt}{27ws^3} = \dot{w} = \dot{z}$. Now $\dot{y}$ and $\dot{z}$ being found, the reſt is perform'd as in the firſt Example.

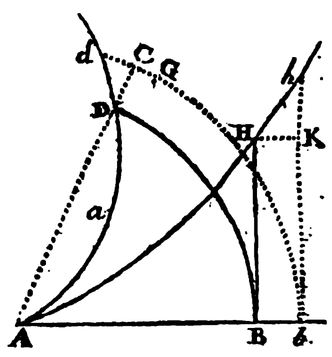
8. Now if from any point Q of a Curve, a perpendicular QV is let fall on MN, and a Curve is to be found, whoſe length may be known from the length which ariſes by applying the Area QRNV to any given Line; let that given Line be call'd E, the length $\frac{QRNV}{E}$ which is produced by ſuch application be call'd v, and its fluxion $\dot{v}$. And ſince the fluxion of the Area QRNV is to the Fluxion of the Area of a rectangular parallelogram made upon VN, with the height E, as the Ordinate or moving line $NR = t$, by which this is deſcribed, to the moving Line E, by which the other is deſcribed in

the ſame time; and the fluxions $\dot{v}$ and $\dot{s}$ of the lines v and MN, (or s,) or of the lengths which ariſe by applying thoſe Areas to the given Line E, are in the ſame ratio; it will be $\dot{v}=\frac{\dot{s}t}{E}$. Therefore by this Rule the value of $\dot{v}$ is to be inquired, and the reſt to be perform'd as in the Examples aforegoing.

9. Ex. 4. Let QR be an Hyperbola which is defined by this Equation, $aa+\frac{ass}{c}=tt$; and thence ariſes (by Prob. 1.) $\frac{as\dot{s}}{c}=t\dot{t}$, or $\frac{as\dot{s}}{ct}=\dot{t}$. Then if for the other two Equations are aſſumed $x=s$ and $y=v$; the firſt will give $1=\dot{s}$, whence $\dot{v}=\frac{\dot{s}t}{E}=\frac{t}{E}$; and the latter will give $\dot{y}=\dot{v}$, or $z=\frac{t}{E}$, then from hence $\dot{z}=\frac{\dot{t}}{E}$, and ſubſtituting $\frac{as\dot{s}}{ct}$ or $\frac{as}{ct}$ for $\dot{t}$, it becomes $\dot{z}=\frac{as}{Ect}$. Now $\dot{y}$ and $\dot{z}$ being found, make $\frac{1-yy}{\dot{z}}=$ CP, and $\dot{y}\times$ CP $=$ PL, as before, and thence the Point C will be determin'd, and the Curve in which all ſuch points are ſituated: The length of which Curve will be known from the length DC, which is equivalent to CP $-\,v$, as is ſufficiently ſhewn before.

10. There is alſo another method, by which the Problem may be reſolved; and that is by finding Curves whoſe fluxions are either equal to the fluxion of the propoſed Curve, or are compounded of the fluxion of that, and of other Lines. And this may ſometimes be of uſe, in converting mechanical Curves into equable Geometrical Curves; of which thing there is a remarkable Example in ſpiral lines.

11. Let AB be a right Line given in poſition, BD an Arch moving upon AB as an Abſciſs, and yet retaining A as its Center, ADd a Spiral, at which that arch is continually terminated, bd an arch indefinitely near it, or the place into which the arch BD by its motion next arrives, DC a perpendicular to the arch bd, dG the difference of the arches, AH another Curve equal to the Spiral AD, BH a right Line moving perpendicularly upon AB, and terminated at the Curve AH, bb the next place into which that right Line moves, and HK perpendicular to bb.



bb. And in the infinitely little triangles DC*d* and HK*b*, ſince DC and HK are equal to the ſame third Line B*b*, and therefore equal to each other, and D*d* and H*b* (by hypotheſis) are correſpondent parts of equal Curves, and therefore equal, as alſo the angles at C and K are right angles; the third ſides *d*C and *b*K will be equal alſo. Moreover ſince it is AB : BD :: A*b* : *b*C :: A*b* — AB (B*b*) : *b*C — BD (CG); therefore $\frac{\text{BD} \times \text{B}b}{\text{AB}} = \text{CG}$. If this be taken away from *d*G, there will remain $d\text{G} - \frac{\text{BD} \times \text{B}b}{\text{AB}} = d\text{C} = b\text{K}$. Call therefore AB $= z$, BD $= v$, and BH $= y$, and their fluxions $\dot{z}$, $\dot{v}$, and $\dot{y}$ reſpectively, ſince B*b*, *d*G, and *b*K are the contemporaneous moments of the ſame, by the acceſſion of which they become A*b*, *bd*, and *bb*, and therefore are to each other as the fluxions. Therefore for the moments in the laſt Equation let the fluxions be ſubſtituted, as alſo the letters for the Lines, and there will ariſe $\dot{v} - \frac{v\dot{z}}{z} = \dot{y}$. Now of theſe fluxions, if $\dot{z}$ be ſuppos'd equable, or the unit to which the reſt are refer'd, the Equation will be $\dot{v} - \frac{v}{z} = \dot{y}$.

12. Wherefore the relation between AB and BD, (or between z and v,) being given by any Equation, by which the Spiral is defined, the fluxion $\dot{v}$ will be given, (by Prob. 1.) and thence alſo the fluxion $\dot{y}$, by putting it equal to $\dot{v} - \frac{v}{z}$. And (by Prob. 2.) this will give the line y, or BH, of which it is the fluxion.

13. Ex. 1. If the Equation $\frac{zz}{a} = v$ were given, which is to the Spiral of *Archimedes*, thence (by Prob. 1.) $\frac{2z}{a} = \dot{v}$. From hence take $\frac{v}{z}$, or $\frac{z}{a}$, and there will remain $\frac{z}{a} = \dot{y}$, and thence (by Prob. 2.) $\frac{zz}{2a} = y$. Which ſhews the Curve AH, to which the Spiral AD is equal, to be the Parabola of *Apollonius*, whoſe Latus rectum is $2a$; or whoſe Ordinate BH is always equal to half the Arch BD.

14. Ex. 2. If the Spiral be propoſed which is defined by the Equation $z^3 = av^2$, or $v = \frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, there ariſes (by Prob. 1.) $\frac{3z^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \dot{v}$, from which if you take $\frac{v}{z}$, or $\frac{z^{\frac{1}{2}}}{a^{\frac{1}{2}}}$, there will remain $\frac{z^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \dot{y}$, and thence (by Prob. 2.) will be produced $\frac{z^{\frac{3}{2}}}{3a^{\frac{1}{2}}} = y$. That is, $\frac{1}{3}$BD $=$ BH, AH being a Parabola of the ſecond kind.

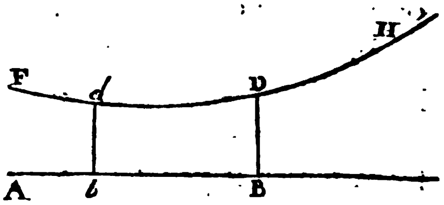
15. Ex. 3. If the Equation to the Spiral be $z\sqrt{\frac{a+z}{c}} = v$, thence (by Prob. 1.) $\frac{2a+3z}{2\sqrt{ac+cz}} = \dot{v}$; from whence if you take away $\frac{v}{z}$ or $\sqrt{\frac{a+z}{c}}$, there will remain $\frac{z}{2\sqrt{ac+cz}} = \dot{y}$. Now ſince the quantity generated by this fluxion $\dot{y}$ cannot be found by Prob. 2. unleſs it be reſolved into an infinite Series; according to the tenor of the Scholium to Prob. 9. I reduce it to the form of the Equations in the firſt column of the Tables, by ſubſtituting z^{η} for z; then it becomes $\frac{z^{2\eta-1}}{2\sqrt{ac+cz^{\eta}}} = \dot{y}$, which Equation belongs to the 2d Species of the 4th Order of Table 1. And by comparing the terms, it is $d = \frac{1}{2}$, $e = ac$, and $f = c$, ſo that $\frac{z-2a}{3c}\sqrt{ac+cz} = t = y$. Which Equation belongs to a Geometrical Curve AH, which is equal in length to the Spiral AD.

PROB. XII.

To determine the Lengths of Curves.

1. In the foregoing Problem we have ſhewn, that the Fluxion of a Curve-line is equal to the ſquare-root of the ſum of the ſquares of the Fluxions of the Abſciſs and of the perpendicular Ordinate. Wherefore if we take the Fluxion of the Abſciſs for an uniform and determinate meaſure, or for an Unit to which the other Fluxions are to be refer'd, and alſo if from the Equation which defines the Curve, we find the Fluxion of the Ordinate, we ſhall have the Fluxion of the Curve-line, from whence (by Problem 2.) its Length may be deduced.

2. Ex. 1. Let the Curve FDH be propoſed, which is defined by the Equation $\frac{z^3}{aa} + \frac{aa}{12z} = y$; making the Abſciſs AB $= z$, and the moving Ordinate DB $= y$. Then from the Equation will be had, (by Prob. 1.) $\frac{3zz}{aa} - \frac{aa}{12zz} = \dot{y}$, the fluxion of z being 1, and $\dot{y}$ being the fluxion of y. Then adding the ſquares of the fluxions, the ſum will be $\frac{9z^4}{a^4} + \frac{1}{2} + \frac{a^4}{144z^4} = \dot{t}\dot{t}$, and extracting the root, $\frac{3zz}{aa} + \frac{aa}{12zz} = \dot{t}$,



$= \dot{t}$, and thence (by Prob. 2.) $\frac{z^3}{aa} - \frac{aa}{12z} = t$. Here $\dot{t}$ ſtands for the fluxion of the Curve, and t for its Length.

3. Therefore if the length dD of any portion of this Curve were required, from the points d and D let fall the perpendiculars db and DB to AB, and in the value of t ſubſtitute the quantities Ab and AB ſeverally for z, and the difference of the reſults will be dD the Length required. As if $Ab = \frac{1}{2}a$, and $AB = a$, writing $\frac{1}{2}a$ for z, it becomes $t = -\frac{a}{24}$; then writing a for z, it becomes $t = \frac{11a}{12}$, from whence if the firſt value be taken away, there will remain $\frac{23a}{24}$ for the length dD. Or if only Ab be determin'd to be $\frac{1}{2}a$, and AB be look'd upon as indefinite, there will remain $\frac{z^3}{aa} - \frac{aa}{12z} + \frac{a}{24}$ for the value of dD.

4. If you would know the portion of the Curve which is repreſented by t, ſuppoſe the value of t to be equal to nothing, and there ariſes $z^4 = \frac{a^4}{12}$, or $z = \frac{a}{\sqrt[4]{12}}$. Therefore if you take $AB = \frac{a}{\sqrt[4]{12}}$, and erect the perpendicular bd, the length of the Arch dD will be t, or $\frac{z^3}{aa} - \frac{aa}{12z}$. And the ſame is to be underſtood of all Curves in general.

5. After the ſame manner by which we have determin'd the length of this Curve, if the Equation $\frac{z^4}{a^3} + \frac{a^3}{32z^2} = y$ be propoſed, for defining the nature of another Curve; there will be deduced $\frac{z^4}{a^3} - \frac{a^3}{32z^2} = t$; or if this Equation be propoſed, $\frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}} - \frac{1}{3}a^{\frac{1}{2}}z^{\frac{1}{2}} = y$, there will ariſe $\frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}} + \frac{1}{3}a^{\frac{1}{2}}z^{\frac{1}{2}} = t$. Or in general, if it is $cz^{\theta} + \frac{z^{2-\theta}}{4\theta\theta c - 8\theta c} = y$, where θ is uſed for repreſenting any number, either Integer or Fraction, we ſhall have $cz^{\theta} - \frac{z^{2-\theta}}{4\theta\theta c - 8\theta c} = t$.

6. Ex. 2. Let the Curve be propoſed which is defined by this Equation $\frac{2aa + 3zz}{3aa}\sqrt{aa + zz} = y$; then (by Prob. 1.) will be had $\dot{y} = \frac{4a^4z + 8a^2z^3 + 4z^5}{3a^4y}$, or exterminating y, $\dot{y} = \frac{2z}{aa}\sqrt{aa + zz}$. To the ſquare of which add 1, and the ſum will be $1 + \frac{4zz}{aa} + \frac{4z^4}{a^4}$,

and its Root $1 + \frac{2zz}{aa} = \dot{t}$. Hence (by Prob. 2.) will be obtain'd $z + \frac{2z^3}{3a^2} = t$.

7. Ex. 3. Let a Parabola of the ſecond kind be propoſed, whoſe Equation is $z^3 = ay^2$, or $\frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}} = y$, and thence by Prob. 1. is derived $\frac{3z^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \dot{y}$. Therefore $\sqrt{1 + \frac{9z}{4a}} = \sqrt{1 + \dot{y}\dot{y}} = \dot{t}$. Now ſince the length of the Curve generated by the Fluxion $\dot{t}$ cannot be found by Prob. 2. without a reduction to an infinite Series of ſimple Terms, I conſult the Tables in Prob. 9. and according to the Scholium belonging to it, I have $t = \frac{8a + 18z}{27}\sqrt{1 + \frac{9z}{4a}}$. And thus you may find the lengths of theſe Parabolas $z^5 = ay^4$, $z^7 = ay^6$, $z^9 = ay^8$, &c.

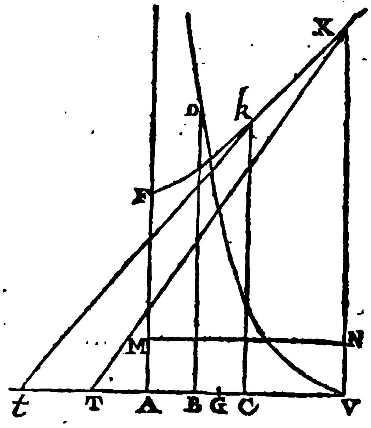
8. Ex. 4. Let the Parabola be propoſed, whoſe Equation is $z^4 = ay^3$, or $\frac{z^{\frac{4}{3}}}{a^{\frac{1}{3}}} = y$; and thence (by Prob. 1.) will ariſe $\frac{4z^{\frac{1}{3}}}{3a^{\frac{1}{3}}} = \dot{y}$. Therefore $\sqrt{1 + \frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}} = \sqrt{\dot{y}\dot{y} + 1} = \dot{t}$. This being found, I conſult the Tables according to the aforeſaid Scholium, and by comparing with the 2d Theorem of the 5th Order of Table 2, I have $z^{\frac{1}{3}} = x$, $\sqrt{1 + \frac{16xx}{9a^{\frac{2}{3}}}} = v$, and $\frac{3}{2}s = t$. Where x denotes the Abſciſs, y the Ordinate, and s the Area of the Hyperbola, and t the length which ariſes by applying the Area $\frac{3}{2}s$ to linear unity.

9. After the ſame manner the lengths of the Parabolas $z^6 = ay^5$, $z^8 = ay^7$, $z^{10} = ay^9$, &c. may alſo be reduced to the Area of the Hyperbola.

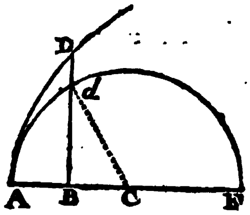
10. Ex. 5. Let the Ciſſoid of the Ancients be propoſed, whoſe Equation is $\frac{aa - 2az + zz}{\sqrt{az - zz}} = y$, and thence (by Prob. 1.) $\frac{-a - 2z}{2zz}\sqrt{az - zz} = \dot{y}$, and therefore $\frac{a}{2z}\sqrt{\frac{a + 3z}{z}} = \sqrt{\dot{y}\dot{y} + 1} = \dot{t}$; which by writing z^{η} for $\frac{1}{z}$ or z^{-1}, becomes $\frac{a}{2z}\sqrt{az^{\eta} + 3} = \dot{t}$, an Equation of the 1ſt Species of the 3d Order of Table 2; then comparing the Terms, it is $\frac{a}{2} = d$, $3 = e$, and $a = f$; ſo that $z = \frac{1}{z^{\eta}} = x^2$, $\sqrt{a + 3xx} = v$, and $6s - \frac{2v^3}{x} = \frac{4de}{\eta f}$ into $\frac{v^3}{2ex} - s = t$.

And taking a for Unity, by the Multiplication or Diviſion of which, theſe Quantities may be reduced to a juſt number of Dimenſions, it becomes $az = xx$, $\sqrt{aa + 3xx} = v$, and $\frac{6s}{a} - \frac{2v^3}{ax} = t$: Which are thus conſtructed.

11. The Ciſſoid being VD, AV the Diameter of the Circle to which it is adapted, AF its Aſymptote, and DB perpendicular to AV, cutting the Curve in D; with the Semiaxis AF = AV, and the Semiparameter AG = $\frac{1}{3}$AV, let the Hyperbola F*k*K be deſcribed; and taking AC a mean Proportional between AB and AV, at C and V let C*k* and VK drawn perpendicular to AV, cut the Hyperbola in *k*, and K, and let right Lines *kt* and KT touch it in thoſe points, and cut AV in *t* and T; and at AV let the Rectangle AVNM be deſcribed, equal to the Space TK*kt*. Then the length of the Ciſſoid VD will be ſextuple of the Altitude VN.

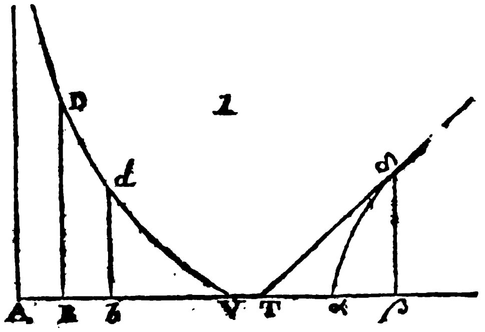


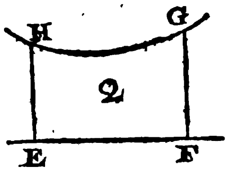
12. Ex. 6. Suppoſing A*d* to be an Ellipſis, which the Equation $\sqrt{az - 2zz} = y$ repreſents; let the mechanical Curve AD be propoſed of ſuch a nature, that if B*d*, or y, be produced till it meets this Curve at D, let BD be equal to the Elliptical Arch A*d*. Now that the length of this may be determin'd, the Equation $\sqrt{az - 2zz} = y$ will give $\frac{a - 4z}{2\sqrt{az - 2zz}} = \dot{y}$, to the ſquare of which if 1 be added, there ariſes $\frac{aa - 4az + 8zz}{4az - 8zz}$, the ſquare of the fluxion of the arch A*d*. To which if 1 be added again, there will ariſe $\frac{aa}{4az - 8zz}$, whoſe ſquare-root $\frac{a}{2\sqrt{az - 2zz}}$ is the fluxion of the Curve-line AD. Where if z be extracted out of the radical, and for z^{-1} be written z^{n}, there will be had $\frac{a}{2z\sqrt{az^{n} - 2}}$, a Fluxion of the 1ſt Species of the 4th Order of Table 2. Therefore the terms being collated, there will ariſe $d = \frac{1}{2}a$, $e = -2$, and $f = a$; ſo that $z = \frac{1}{z^{n}} = x$, $\sqrt{ax - 2xx} = v$, and $\frac{8s}{a} - \frac{4xv}{a} + v = \frac{8de}{nff}$ into $s - \frac{1}{2}xv - \frac{fv}{4e} = t$.

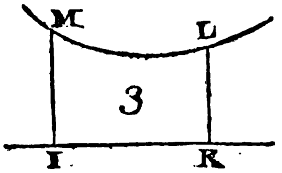


13. The Conſtruction of which is thus; that the right line dC being drawn to the center of the Ellipſis, a parallelogram may be made upon AC, equal to the ſector ACd, and the double of its height will be the length of the Curve AD.

14. Ex. 7. Making $A\beta = \varphi$, (Fig. 1.) and $\alpha\delta$ being an Hyperbola, whoſe Equation is $\sqrt{-a+b\varphi\varphi} = \beta\delta$, and its tangent δT being drawn; let the Curve VdD be propoſed, whoſe Abſciſs is $\frac{1}{\varphi\varphi}$, and its perpendicular Ordinate is the length BD, which ariſes by applying the Area $\alpha\delta T\alpha$ to linear unity. Now that the length of this Curve VD may be determin'd, I ſeek the fluxion of the Area $\alpha\delta T\alpha$, when AB flows uniformly, and I find it to be $\frac{a}{4bz}\sqrt{b-az}$, putting $AB = z$, and its fluxion unity. For 'tis $AT = \frac{a}{b\varphi} = \frac{a}{b}\sqrt{z}$, and its fluxion is $\frac{a}{2b\sqrt{z}}$, whoſe half drawn into the altitude $\beta\delta$, or $\sqrt{-a+\frac{b}{z}}$, is the fluxion of the Area $\alpha\delta$T, deſcribed by the Tangent δT. Therefore that fluxion is $\frac{a}{4bz}\sqrt{b-az}$, and this apply'd to unity becomes the fluxion of the Ordinate BD. To the ſquare of this $\frac{aab-a^3z}{16b^2z^2}$ add 1, the ſquare of the fluxion BD, and there ariſes $\frac{aab-a^3z+16b^2z^2}{16b^2z^2}$, whoſe root $\frac{1}{4bz}\sqrt{a^2b-a^3z+16b^2z^2}$ is the fluxion of the Curve VD. But this is a fluxion of the 1ſt Species of the 7th Order of Table 2: and the terms being collated, there will be $\frac{1}{4b} = d$, $aab = e$, $-a^3 = f$, $16b^2 = g$, and therefore $z = x$, and $\sqrt{a^2b-a^3x+16b^2x^2} = v$, (an Equation to one Conic Section, ſuppoſe HG, (Fig. 2.) whoſe Area EFGH is s, where $EF = x$, and $FG = v$;) alſo $\frac{1}{z} = \xi$, and $\sqrt{16bb-a^3\xi+ab\xi^2} = \Upsilon$, (an Equation to another Conic Section,







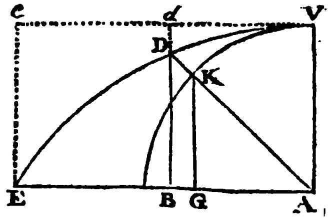
Section, ſuppoſe ML (Fig. 3.) whoſe Area IKLM is σ, where IK $= \xi$, and KL $= \Upsilon$;) Laſtly $\frac{2aabb\xi\Upsilon - a^3b\Upsilon - a^4v - 4aabb\sigma - 32abbs}{64b^4 - a^4} = t$,

15. Wherefore that the length of any portion D*d* of the Curve VD may be known, let fall *db* perpendicular to AB, and make A*b* $= z$, and thence, by what is now found, ſeek the value of t. Then make AB $= z$, and thence alſo ſeek for t. And the difference of theſe two values of t will be the length D*d* required.

16. Ex. 8. Let the Hyperbola be propos'd, whoſe Equation is $\sqrt{aa+bzz} = y$, and thence, (by Prob. 1.) will be had $\dot{y} = \frac{bz}{y}$, or $\frac{bz}{\sqrt{aa+bzz}}$. To the ſquare of this add 1, and the root of the ſum will be $\sqrt{\frac{aa+bzz+bbzz}{aa+bzz}} = \dot{t}$. Now as this fluxion is not to be found in the Tables, I reduce it to an infinite Series; and firſt by diviſion it becomes $\dot{t} = \sqrt{1+\frac{b^2}{a^2}z^2-\frac{b^3}{a^4}z^4+\frac{b^4}{a^6}z^6-\frac{b^5}{a^8}z^8}$, &c. and extracting the root, $\dot{t} = 1+\frac{b^2}{2a^2}z^2-\frac{4b^3+b^4}{8a^4}z^4+\frac{8b^4+4b^5+b^6}{16a^6}z^6$, &c. And hence (by Prob. 2.) may be had the length of the Hyperbolical Arch, or $t = z+\frac{b^2}{6a^2}z^3-\frac{4b^3+b^4}{40a^4}z^5+\frac{8b^4+4b^5+b^6}{112a^6}z^7$, &c.

17. If the Ellipſis $\sqrt{aa-bzz} = y$ were propoſed, the Sign of b ought to be every where changed, and there will be had $z+\frac{b^2}{6a^2}z^3+\frac{4b^3-b^4}{40a^4}z^5+\frac{8b^4-4b^5+b^6}{112a^6}z^7$, &c. for the length of its Arch. And likewiſe putting Unity for b, it will be $z+\frac{z^3}{6a^2}+\frac{3z^5}{40a^4}+\frac{5z^7}{112a^6}$, &c. for the length of the Circular Arch. Now the numeral coefficients of this ſeries may be found *ad infinitum*, by multiplying continually the terms of this Progreſſion $\frac{1\times1}{2\times3}$, $\frac{3\times3}{4\times5}$, $\frac{5\times5}{6\times7}$, $\frac{7\times7}{8\times9}$, $\frac{9\times9}{10\times11}$, &c.

18. Ex. 9. Laſtly, let the Quadratrix VDE be propoſed, whoſe Vertex is V, A being the Center, and AV the ſemidiameter of the interior Circle, to which it is adapted, and the Angle VAE being a right Angle. Now any right Line AKD being drawn through A, cutting the Circle in K, and the Quadratrix in D, and the perpendiculars KG, DB being let fall to AE; call AV $= a$, AG $= z$, VK $= x$, and BD $= y$, and it



will be as in the foregoing Example, $x = z + \frac{z^3}{6a^2} + \frac{3z^5}{40a^4} + \frac{5z^7}{112a^6}$, &c. Extract the root z, and there will ariſe $z = x - \frac{x^3}{6a^2} + \frac{x^5}{120a^4} - \frac{x^7}{5040a^6}$, &c. whoſe Square ſubtract from AKq, or a^2, and the root of the remainder $a - \frac{x^2}{2a} + \frac{x^4}{24a^3} - \frac{x^6}{720a^5}$, &c. will be GK. Now whereas by the nature of the Quadratrix 'tis $AB = VR = x$, and ſince it is $AG : GK :: AB : BD\ (y)$, divide $AB \times GK$ by AG, and there will ariſe $y = a - \frac{xx}{3a} - \frac{x^4}{45a^3} - \frac{2x^6}{945a^5}$, &c. And thence, (by Prob. 1.) $\dot{y} = -\frac{2x}{3a} - \frac{4x^3}{45a^3} - \frac{4x^5}{315a^5}$, &c. to the ſquare of which add 1, and the root of the ſum will be $1 + \frac{2xx}{9aa} + \frac{14x^4}{405a^4} + \frac{604x^6}{127575a^6}$, &c. $= \dot{t}$. Whence (by Prob. 2.) t may be obtain'd, or the Arch of the Quadratrix; *viz.* $VD = x + \frac{2x^3}{27a^2} + \frac{14x^5}{2825a^4} + \frac{604x^7}{893025a^6}$, &c.

THE

THE METHOD of FLUXIONS AND INFINITE SERIES.

ANNOTATIONS on the Introduction: OR, The Reſolution of Equations by INFINITE SERIES.

SECT. I. *Of the Nature and Conſtruction of Infinite or Converging Series.*

1. THE great Author of the foregoing Work begins it with a ſhort Preface, in which he lays down his main deſign very conciſely. He is not to be here underſtood, as if he would reproach the modern Geometricians with deſerting the Ancients, or with abandoning their Synthetical Method of Demonſtration, much leſs that he intended to diſparage the Analytical Art; for on the contrary he has very much improved both Methods, and particularly in this Treatiſe he wholly applies himſelf to cultivate Analyticks, in which he has ſucceeded to univerſal applauſe and admiration. Not but that we ſhall find here ſome examples of the Synthetical Method likewiſe, which are very maſterly and elegant. Almoſt all that remains of the ancient Geometry is indeed Synthetical, and proceeds by way of demonſtrating truths already known, by ſhewing their dependence upon the Axioms, and

other firſt Principles, either mediately or immediately. But the buſineſs of Analyticks is to inveſtigate ſuch Mathematical Truths as really are, or may be ſuppos'd at leaſt to be unknown. It aſſumes thoſe Truths as granted, and argues from them in a general manner, till after a ſeries of argumentation, in which the ſeveral ſteps have a neceſſary connexion with each other, it arrives at the knowledge of the propoſition required, by comparing it with ſomething really known or given. This therefore being the Art of Invention, it certainly deſerves to be cultivated with the utmoſt induſtry. Many of our modern Geometricians have been perſuaded, by conſidering the intricate and labour'd Demonſtrations of the Ancients, that they were Maſters of an Analyſis purely Geometrical, which they ſtudiouſly conceal'd, and by the help of which they deduced, in a direct and ſcientifical manner, thoſe abſtruſe Propoſitions we ſo much admire in ſome of their writings, and which they afterwards demonſtrated Synthetically. But however this may be, the loſs of that Analyſis, if any ſuch there were, is amply compenſated, I think, by our preſent Arithmetical or Algebraical Analyſis, eſpecially as it is now improved, I might ſay perfected, by our ſagacious Author in the Method before us. It is not only render'd vaſtly more univerſal and extenſive than that other in all probability could ever be, but is likewiſe a moſt compendious Analyſis for the more abſtruſe Geometrical Speculations, and for deriving Conſtructions and Synthetical Demonſtrations from thence; as may abundantly appear from the enſuing Treatiſe.

2. The conformity or correſpondence, which our Author takes notice of here, between his new-invented Doctrine of infinite Series, and the commonly received Decimal Arithmetick, is a matter of conſiderable importance, and well deſerves, I think, to be ſet in a fuller Light, for the mutual illuſtration of both; which therefore I ſhall here attempt to perform. For Novices in this Doctrine, tho' they may already be well acquainted with the Vulgar Arithmetick, and with the Rudiments of the common Algebra, yet are apt to apprehend ſomething abſtruſe and difficult in infinite Series; whereas indeed they have the ſame general foundation as Decimal Arithmetick, eſpecially Decimal Fractions, and the ſame Notion or Notation is only carry'd ſtill farther, and render'd more univerſal. But to ſhew this in ſome kind of order, I muſt inquire into theſe following particulars. Firſt I muſt ſhew what is the true Nature, and what are the genuine Principles, of our common Scale of Decimal Arithmetick. Secondly what is the nature of other particular Scales, which have been, or

may be, occaſionally introduced. Thirdly, what is the nature of a general Scale, which lays the foundation for the Doctrine of infinite Series. Laſtly, I ſhall add a word or two concerning that Scale of Arithmetick in which the Root is unknown, and therefore propoſed to be found; which gives occaſion to the Doctrine of Affected Equations.

Firſt then as to the common Scale of Decimal Arithmetick, it is that ingenious Artifice of expreſſing, in a regular manner, all conceivable Numbers, whether Integers or Fractions, Rational or Surd, by the ſeveral Powers of the number *Ten*, and their Reciprocals; with the aſſiſtance of other ſmall Integer Numbers, not exceeding Nine, which are the Coefficients of thoſe Powers. So that Ten is here the Root of the Scale, which if we denote by the Character X, as in the *Roman* Notation, and its ſeveral Powers by the help of this Root and Numeral Indexes, ($X^1 = 10$, $X^2 = 100$, $X^3 = 1000$, $X^4 = 10000$, &c.) as is uſual; then by aſſuming the Coefficients 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, as occaſion ſhall require, we may form or expreſs any Number in this Scale. Thus for inſtance $5X^4 + 7X^3 + 4X^2 + 8X^1 + 3X^0$ will be a particular Number expreſs'd by this Scale, and is the ſame as 57483 in the common way of Notation. Where we may obſerve, that this laſt differs from the other way of Notation only in this, that here the ſeveral Powers of X (or Ten) are ſuppreſs'd, together with the Sign of Addition $+$, and are left to be ſupply'd by the Underſtanding. For as thoſe Powers aſcend regularly from the place of Units, (in which is always X^0, or 1, multiply'd by its Coefficient, which here is 3,) the ſeveral Powers will eaſily be underſtood, and may therefore be omitted, and the Coefficients only need to be ſet down in their proper order. Thus the Number 7906538 will ſtand for $7X^6 + 9X^5 + 0X^4 + 6X^3 + 5X^2 + 3X^1 + 8X^0$, when you ſupply all that is underſtood. And the Number 1736 (by ſuppreſſing what may be eaſily underſtood,) will be equivalent to $X^3 + 7X^2 + 3X + 6$; and the like of all other Integer Numbers whatever, expreſs'd by this Scale, or with this Root X, or Ten.

The ſame Artifice is uniformly carry'd on, for the expreſſing of all Decimal Fractions, by means of the Reciprocals of the ſeveral Powers of Ten, ſuch as $\frac{1}{X} = 0,1$; $\frac{1}{X^2} = 0,01$; $\frac{1}{X^3} = 0,001$; &c. which Reciprocals may be intimated by negative Indices. Thus the Decimal Fraction 0,3172 ſtands for $3X^{-1} + 1X^{-2} + 7X \quad + 2X^{-4}$; and the mixt Number 526,384 (by ſupplying what is underſtood) becomes $5X^2 + 2X^1 + 6X^0 + 3X^{-1} + 8X^{-2} + 4X^{-3}$; and the infinite or interminate Decimal Fraction 0,9999999, &c. ſtands for $9X^{-1} + 9X^{-2} + 9X^{-3} + 9X^{-4} + 9X^{-5} + 9X^{-6}$, &c, which infinite Series is equivalent to Unity. So that by this Decimal Scale, (or by the ſeveral Powers of Ten and their Reciprocals, together with their Coefficients, which are all the whole Numbers below Ten,) all conceivable Numbers may be expreſs'd, whether they are integer or fracted, rational or irrational; at leaſt by admitting of a continual progreſs or approximation *ad infinitum.*

And the like may be done by any other Scale, as well as the Decimal Scale, or by admitting any other Number, beſides Ten, to be the Root of our Arithmetick. For the Root Ten was an arbitrary Number, and was at firſt aſſumed by chance, without any previous conſideration of the nature of the thing. Other Numbers perhaps may be aſſign'd, which would have been more convenient, and which have a better claim for being the Root of the Vulgar Scale of Arithmetick. But however this may prevail in common affairs, Mathematicians make frequent uſe of other Scales; and therefore in the ſecond place I ſhall mention ſome other particular Scales, which have been occaſionally introduced into Computations.

The moſt remarkable of theſe is the Sexagenary or Sexageſimal Scale of Arithmetick, of frequent uſe among Aſtronomers, which expreſſes all poſſible Numbers, Integers or Fractions, Rational or Surd, by the Powers of *Sixty,* and certain numeral Coefficients not exceeding fifty-nine. Theſe Coefficients, for want of peculiar Characters to repreſent them, muſt be expreſs'd in the ordinary Decimal Scale. Thus if ξ ſtands for 60, as in the *Greek* Notation, then one of theſe Numbers will be $53\xi^2 + 9\xi^1 + 34\xi^0$, or in the Sexagenary Scale $53''$, $9'$, 34°, which is equivalent to 191374° in the Decimal Scale. Again, the Sexageſimal Fraction 53°, $9'$, $34''$, will be the ſame as $53\xi^0 + 9\xi^{-1} + 34\xi^{-2}$, which in Decimal Numbers will be 53,159444, &c. *ad infinitum.* Whence it appears by the way, that ſome Numbers may be expreſs'd by a finite number of Terms in one Scale, which in another cannot be expreſs'd but by approximation, or by a progreſſion of Terms *in infinitum.*

Another particular Scale that has been conſider'd, and in ſome meaſure has been admitted into practice, is the Duodecimal Scale, which expreſſes all Numbers by the Powers of *Twelve.* So in common affairs we ſay a Dozen, a Dozen of Dozens or a Groſs, a Dozen of Groſſes or a great Groſs, *&c.* And this perhaps would have been the moſt convenient Root of all others, by the Powers of which

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to conſtruct the popular Scale of Arithmetick; as not being ſo big but that its Multiples, and all below it, might be eaſily committed to memory; and it admits of a greater variety of Diviſors than any Number not much greater than itſelf. Beſides, it is not ſo ſmall, but that Numbers expreſs'd hereby would ſufficiently converge, or by a few figures would arrive near enough to the Number required; the contrary of which is an inconvenience, that muſt neceſſarily attend the taking too ſmall a Number for the Root. And to admit this Scale into practice, only two ſingle Characters would be wanting, to denote the Coefficients Ten and Eleven.

Some have conſider'd the Binary Arithmetick, or that Scale in which *Two* is the Root, and have pretended to make Computations by it, and to find conſiderable advantages in it. But this can never be a convenient Scale to manage and expreſs large Numbers by, becauſe the Root, and conſequently its Powers, are ſo very ſmall, that they make no diſpatch in Computations, or converge exceeding ſlowly. The only Coefficients that are here neceſſary are 0 and 1. Thus $1\times2^5+1\times2^4+0\times2^3+1\times2^2+1\times2^1+0\times2^0$ is one of theſe Numbers, (or compendiouſly 110110,) which in the common Notation is no more than 54. Mr. *Leibnits* imagin'd he had found great Myſteries in this Scale. See the Memoirs of the Royal Academy of *Paris*, *Anno* 1703.

In common affairs we have frequent recourſe, though tacitly, to Millenary Arithmetick, and other Scales, whoſe Roots are certain Powers of Ten. As when a large Number, for the convenience of reading, is diſtinguiſh'd into Periods of three figures: As 382,735,628,490. Here 382, and 735, *&c.* may be conſider'd as Coefficients, and the Root of the Scale is 1000. So when we reckon by Millions, Billions, Trillions, *&c.* a Million may be conceived as the Root of our Arithmetick. Alſo when we divide a Number into pairs of figures, for the Extraction of the Square-root; into ternaries of figures for the Extraction of the Cube-root; *&c.* we take new Scales in effect, whoſe Roots are 100, 1000, *&c.*

Any Number whatever, whether Integer or Fraction, may be made the Root of a particular Scale, and all conceivable Numbers may be expreſs'd or computed by that Scale, admitting only of integral and affirmative Coefficients, whoſe number (including the Cypher 0) need not be greater than the Root. Thus in Quinary Arithmetick, in which the Scale is compoſed of the Powers of the Root 5, the Coefficients need be only the five Numbers 0, 1, 2, 3, 4, and yet all Numbers whatever are expreſſible by this Scale, at leaſt by approximation,

mation, to what accuracy we pleaſe. Thus the common Number 2827,92 in this Arithmetick would be $4 \times 5^4 + 2 \times 5^3 + 3 \times 5^2 + 0 \times 5^1 + 2 \times 5^0 + 4 \times 5^{-1} + 3 \times 5^{-2}$; or if we may ſupply the ſeveral Powers of 5 by the Imagination only, as we do thoſe of Ten in the common Scale, this Number will be 42302,43 in Quinary Arithmetick.

All vulgar Fractions and mixt Numbers are, in ſome meaſure, the expreſſing of Numbers by a particular Scale, or making the Denominator of the Fraction to be the Root of a new Scale. Thus $\frac{2}{3}$ is in effect $0 \times 3^0 + 2 \times 3^{-1}$; and $8\frac{3}{5}$ is the ſame as $8 \times 5^0 + 3 \times 5^{-1}$; and $25\frac{4}{9}$ reduced to this Notation will be $25 \times 9^0 + 4 \times 9^{-1}$, or rather $2 \times 9^1 + 7 \times 9^0 + 4 \times 9^{-1}$. And ſo of all other Fractions and mixt Numbers.

A Number computed by any one of theſe Scales is eaſily reduced to any other Scale aſſign'd, by ſubſtituting inſtead of the Root in one Scale, what is equivalent to it expreſs'd by the Root of the other Scale. Thus to reduce Sexagenary Numbers to Decimals, becauſe $60 = 6 \times 10$, or $\xi = 6X$, and therefore $\xi^2 = 36X^2$, $\xi^3 = 216X^3$, *&c.* by the ſubſtitution of theſe you will eaſily find the equivalent Decimal Number. And the like in all other Scales.

The Coefficients in theſe Scales are not neceſſarily confin'd to be affirmative integer Numbers leſs than the Root, (tho' they ſhould be ſuch if we would have the Scale to be regular,) but as occaſion may require they may be any Numbers whatever, affirmative or negative, integers or fractions. And indeed they generally come out promiſcuouſly in the Solution of Problems. Nor is it neceſſary that the Indices of the Powers ſhould be always integral Numbers, but may be any regular Arithmetical Progreſſion whatever, and the Powers themſelves either rational or irrational. And thus (thirdly) we are come by degrees to the Notion of what is call'd an univerſal Series, or an indefinite or infinite Series. For ſuppoſing the Root of the Scale to be indefinite, or a general Number, which may therefore be repreſented by x, or y, &c. and aſſuming the general Coefficients a, b, c, d, &c. which are Integers or Fractions, affirmative or negative, as it may happen; we may form ſuch a Series as this, $ax^4 + bx^3 + cx^2 + dx^1 + ex^0$, which will repreſent ſome certain Number, expreſs'd by the Scale whoſe Root is x. If ſuch a Number proceeds *in infinitum*, then it is truly and properly call'd an Infinite Series, or a Converging Series, x being then ſuppos'd greater than Unity. Such for example is $x + \frac{1}{2}x^{-1} + \frac{1}{3}x^{-2} + \frac{1}{4}x^{-3}$, &c. where the reſt of the Terms are underſtood *ad infinitum*, and are inſinuated by, *&c.* And it may have any deſcending Arithmetical Progreſſion for its Indices, as $x^{m} - \frac{2}{3}x^{m-1} + \frac{3}{4}x^{m-2} + \frac{4}{5}x^{m-3}$, *&c.*

And thus we have been led by proper gradations, (that is, by arguing from what is well known and commonly received, to what before appear'd to be difficult and obſcure,) to the knowledge of infinite Series, of which the Learner will find frequent Examples in the ſequel of this Treatiſe. And from hence it will be eaſy to make the following general Inferences, and others of a like nature, which will be of good uſe in the farther knowledge and practice of theſe Series; *viz.* That the firſt Term of every regular Series is always the moſt conſiderable, or that which approaches nearer to the Number intended, (denoted by the Aggregate of the Series,) than any other ſingle Term: That the ſecond is next in value, and ſo on: That therefore the Terms of the Series ought always to be diſpoſed in this regular deſcending order, as is often inculcated by our Author: That when there is a Progreſſion of ſuch Terms *in infinitum*, a few of the firſt Terms, or thoſe at the beginning of the Series, are or ſhould be a ſufficient Approximation to the whole; and that theſe may come as near to the truth as you pleaſe, by taking in ſtill more Terms: That the ſame Number in which one Scale may be expreſs'd by a finite number of Terms, in another cannot be expreſs'd but by an infinite Series, or by approximation only, and *vice verſâ*: That the bigger the Root of the Scale is, by ſo much the faſter, *cæteris paribus*, the Series will converge; for then the Reciprocals of the Powers will be ſo much the leſs, and therefore may the more ſafely be neglected: That if a Series converges by increaſing Powers, ſuch as $ax + bx^{2} + cx^{3} + dx^{4}$, &c. the Root x of the Scale muſt be underſtood to be a proper Fraction, the leſſer the better. Yet whenever a Series can be made to converge by the Reciprocals of Ten, or its Compounds, it will be more convenient than a Series that converges faſter; becauſe it will more eaſily acquire the form of the Decimal Scale, to which, in particular Caſes, all Series are to be ultimately reduced. Laſtly, from ſuch general Series as theſe, which are commonly the reſult in the higher Problems, we muſt paſs (by ſubſtitution) to particular Scales or Series, and thoſe are finally to be reduced to the Decimal Scale. And the Art of finding ſuch general Series, and then their Reduction to particular Scales, and laſt of all to the common Scale of Decimal Numbers, is almoſt the whole of the abſtruſer parts of Analyticks, as may be ſeen in a good meaſure by the preſent Treatiſe.

I

I took notice in the fourth place, that this Doctrine of Scales, and Series, gives us an eaſy notion of the nature of affected Equations, or ſhews us how they ſtand related to ſuch Scales of Numbers. In the other Inſtances of particular Scales, and even of general ones, the Root of the Scale, the Coefficients, and the Indices, are all ſuppos'd to be given, or known, in order to find the Aggregate of the Series, which is here the thing required. But in affected Equations, on the contrary, the Aggregate and the reſt are known, and the Root of the Scale, by which the Number is computed, is unknown and required. Thus in the affected Equation $5x^4 + 3x^3 + 0x^2 + 7x = 53070$, the Aggregate of the Series is given, *viz.* the Number 53070, to find x the Root of the Scale. This is eaſily diſcern'd to be 10, or to be a Number expreſs'd by the common Decimal Scale, eſpecially if we ſupply the ſeveral Powers of 10, where they are underſtood in the Aggregate, thus $5X^4 + 3X^3 + 0X^2 + 7X^1 + 0X^0 = 53070$. Whence by compariſon 'tis $x = X = 10$. But this will not be ſo eaſily perceived in other inſtances. As if I had the Equation $4x^4 + 2x^3 + 3x^2 + 0x^1 + 2x^0 + 4x^{-1} + 3x^{-2} = 2827,92$ I ſhould not ſo eaſily perceive that the Root x was 5, or that this is a Number expreſs'd by Quinary Arithmetick, except I could reduce it to this form, $4 \times 5^4 + 2 \times 5^3 + 3 \times 5^2 + 0 \times 5^1 + 2 \times 5^0 + 4 \times 5^{-1} + 3 \times 5^{-2} = 2827,92$, when by compariſon it would preſently appear, that the Root ſought muſt be 5. So that finding the Root of an affected Equation is nothing elſe, but finding what Scale in Arithmetick that Number is computed by, whoſe Reſult or Aggregate is given in the common Scale; which is a Problem of great uſe and extent in all parts of the Mathematicks. How this is to be done, either in Numeral, Algebraical, or Fluxional Equations, our Author will inſtruct us in its due place.

Before I diſmiſs this copious and uſeful Subject of Arithmetical Scales, I ſhall here make this farther Obſervation; that as all conceivable Numbers whatever may be expreſs'd by any one of theſe Scales, or by help of an Aggregate or Series of Powers derived from any Root; ſo likewiſe any Number whatever may be expreſs'd by ſome ſingle Power of the ſame Root, by aſſuming a proper Index, integer or fracted, affirmative or negative, as occaſion ſhall require. Thus in the Decimal Scale, the Root of which is 10, or X, not only the Numbers 1, 10, 100, 1000, *&c.* or 1, 0.1, 0.01, 0.001, *&c.* that is, the ſeveral integral Powers of 10 and their Reciprocals, may be expreſs'd by the ſingle Powers of X or 10, *viz.* X^0, X^1, X^2, X^3, *&c.* or X^0, X^{-1}, X^{-2}, X^{-3}, *&c.* reſpectively, but alſo all the intermediate Numbers, as 2, 3, 4, *&c.* 11, 12, 13, *&c.* may be expreſs'd by ſuch ſingle Powers of X or 10, if we aſſume proper Indices. Thus $2 = X^{0,30103, \&c.}$, $3 = X^{0,47712, \&c.}$ $4 = X^{0,60206, \&c.}$ &c. or $11 = X^{1,04139, \&c.}$ $12 = X^{1,07918, \&c.}$ $456 = X^{2,65896, \&c.}$ And the like of all other Numbers. Theſe Indices are uſually call'd the Logarithms of the Numbers (or Powers) to which they belong, and are ſo many Ordinal Numbers, declaring what Power (in order or ſucceſſion) any given Number is, of any Root aſſign'd: And different Scales of Logarithms will be form'd, by aſſuming different Roots of thoſe Scales. But how theſe Indices, Logarithms, or Ordinal Numbers may be conveniently found, our Author will likewiſe inform us hereafter. All that I intended here was to give a general Notion of them, and to ſhew their dependance on, and connexion with, the ſeveral Arithmetical Scales before deſcribed.

It is eaſy to obſerve from the *Arenarius* of *Archimedes*, that he had fully conſider'd and diſcuſs'd this Subject of Arithmetical Scales, in a particular Treatiſe which he there quotes, by the name of his *ἀρχαὶ*, or Principles; in which (as it there appears) he had laid the foundation of an Arithmetick of a like nature, and of as large an extent, as any of the Scales now in uſe, even the moſt univerſal. It appears likewiſe, that he had acquired a very general notion of the Doctrine and Uſe of Indices alſo. But how far he had accommodated an Algorithm, or Method of Operation, to thoſe his Principles, muſt remain uncertain till that Book can be recover'd, which is a thing more to be wiſh'd than expected. However it may be fairly concluded from his great Genius and Capacity, that ſince he thought fit to treat on this Subject, the progreſs he had made in it was very conſiderable.

But before we proceed to explain our Author's methods of Operation with infinite Series, it may be expedient to enlarge a little farther upon their nature and formation, and to make ſome general Reflexions on their Convergency, and other circumſtances. Now their formation will be beſt explain'd by continual Multiplication after the following manner.

Let the quantity $a + bx + cx^2 + dx^3 + ex^4$, &c. be aſſumed as a Multiplier, conſiſting either of a finite or an infinite number of Terms; and let alſo $\frac{p}{q} + x = 0$ be ſuch a Multiplier, as will give the Root $x = -\frac{p}{q}$. If theſe two are multiply'd together, they will produce $\frac{ap}{q} + \frac{bp+aq}{q}x + \frac{cp+bq}{q}x^2 + \frac{dp+cq}{q}x^3 + \frac{ep+dq}{q}x^4$, &c. $= 0$;

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$=0$; and if inſtead of x we here ſubſtitute its value $-\frac{p}{q}$, the Series will become $\frac{ap}{q}-\frac{bp+aq}{q}\times\frac{p}{q}+\frac{cp+bq}{q}\times\frac{p^2}{q^2}-\frac{dp+cq}{q}\times\frac{p^3}{q^3}+\frac{ep+dq}{q}\times\frac{p^4}{q^4}$, &c. $=0$; or if we divide by $\frac{p}{q}$, and tranſpoſe, it will be $\frac{bp+aq}{q}-\frac{cp+bq}{q}\times\frac{p}{q}+\frac{dp+cq}{q}\times\frac{p^2}{q^2}-\frac{ep+dq}{q}\times\frac{p^3}{q^3}$, &c. $=a$: which Series, thus derived, may give us a good inſight into the nature of infinite Series in general. For it is plain that this Series, (even though it were continued to infinity,) muſt always be equal to a, whatever may be ſuppoſed to be the values of p, q, a, b, c, d, &c. For $\frac{bp}{q}$, the firſt part of the firſt Term, will always be removed or deſtroy'd by its equal with a contrary Sign, in the ſecond part of the ſecond Term. And $\frac{cp}{q}\times\frac{p}{q}$, the firſt part of the ſecond Term, will be removed by its equal with a contrary Sign, in the ſecond part of the third Term, and ſo on: So as finally to leave $\frac{aq}{q}$, or a, for the Aggregate of the whole Series. And here it is likewiſe to be obſerv'd, that we may ſtop whenever we pleaſe, and yet the Equation will be good, provided we take in the *Supplement*, or a due part of the next Term. And this will always obtain, whatever the nature of the Series may be, or whether it be converging or diverging. If the Series be diverging, or if the Terms continually increaſe in value, then there is a neceſſity of taking in that Supplement, to preſerve the integrity of the Equation. But if the Series be converging, or if the Terms continually decreaſe in any compound Ratio, and therefore finally vaniſh or approach to nothing; the Supplement may be ſafely neglected, as vaniſhing alſo, and any number of Terms may be taken, the more the better, as an Approximation to the Quantity a. And thus from a due conſideration of this fictitious Series, the nature of all converging or diverging Series may eaſily be apprehended. Diverging Series indeed, unleſs when the afore-mention'd increaſing Supplement can be aſſign'd and taken in, will be of no ſervice. And this Supplement, in Series that commonly occur, will be generally ſo entangled and complicated with the Coefficients of the Terms of the Series, that altho' it is always to be underſtood, nevertheleſs, it is often impoſſible to be extricated and aſſign'd. But however, converging Series will always be of excellent uſe, as affording a convenient Approximation to the quantity required, when it cannot be otherwiſe exhibited. In theſe the Supplement aforeſaid,

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tho' generally inextricable and unaſſignable, yet continually decreaſes along with the Terms of the Series, and finally becomes leſs than any aſſignable Quantity.

The ſame Quantity may often be exhibited or expreſs'd by ſeveral converging Series; but that Series is to be moſt eſteem'd that has the greateſt Rate of Convergency. The foregoing Series will converge ſo much the faſter, *cæteris paribus*, as p is leſs than q, or as the Fraction $\frac{p}{q}$ is leſs than Unity. For if it be equal to, or greater than Unity, it may become a diverging Series, and will diverge ſo much the faſter, as p is greater than q. The Coefficients will contribute little or nothing to this Convergency or Divergency, if they are ſuppos'd to increaſe or decreaſe (as is generally the caſe) rather in a ſimple and Arithmetical, than a compound and Geometrical Proportion. To make ſome Eſtimate of the Rate of Convergency in this Series, and by analogy in any other of this kind, let k and l repreſent two Terms indefinitely, which immediately ſucceed each other in the progreſſion of the Coefficients of the Multiplier $a + bx + cx^2 + dx^3$, &c. and let the number n repreſent the order or place of k. Then any Term of the Series indefinitely may be repreſented by $\pm \frac{lp + kq}{q^n} p^{n-1}$; where the Sign muſt be $+$ or $-$, according as n is an odd or an even Number. Thus if $n = 1$, then $k = a$, $l = b$, and the firſt Term will be $+ \frac{bp + aq}{q}$. If $n = 2$, then $k = b$, $l = c$, and the ſecond Term will be $- \frac{cp + bq}{q^2} p$. And ſo of the reſt. Alſo if m be the next Term in the aforeſaid progreſſion after l, then $\pm \frac{lp + kq}{q^n} p^{n-1} \mp \frac{mp + lq}{q^{n+1}} p^n$ will be any two ſucceſſive Terms in the ſame Series. Now in order to a due Convergency, the former Term abſolutely conſider'd, that is ſetting aſide the Signs, ſhould be as much greater than the ſucceeding Term, as conveniently may be. Let us ſuppoſe therefore that $\frac{lp + kq}{q^n} p^{n-1}$ is greater than $\frac{mp + lq}{q^{n+1}} p^n$, or (dividing all by the common factor $\frac{p^n}{q^n}$,) that $\frac{lp + kq}{p}$ is greater than $\frac{mp + lq}{q}$, or (multiplying both by pq,) that $lpq + kq^2$ is greater than $mp^2 + lpq$, or (taking away the common lpq,) that kq^2 is greater than mp^2, or (by a farther Diviſion,) that $\frac{k}{m} \times \frac{q^2}{p^2}$ is greater than unity; and as much greater as may be.

This will take effect on a double account; firſt, the greater k is in reſpect of m, and ſecondly, the greater q^2 is in reſpect of p^2. Now in the Multiplier $a+bx+cx^2+dx^3$, &c. if the Coefficients a, b, c, &c. are in any decreaſing Progreſſion, then k will be greater than l, which is greater than m; ſo that *à fortiori* k will be greater than m. Alſo if q be greater than p, and therefore (in a duplicate ratio) q^2 will be greater than p^2. So that (*cæteris paribus*) the degree of Convergency is here to be eſtimated, from the Rate according to which the Coefficients a, b, c, &c. continually decreaſe, compounded with the Ratio, (or rather its duplicate,) according to which q ſhall be ſuppos'd to be greater than p.

The ſame things obtaining as before, the Term $\mp\frac{lp^n}{q^n}$ will be what was call'd the *Supplement* of the Series. For if the Series be continued to a number of Terms denominated by n, then inſtead of all the reſt of the Terms *in infinitum*, we may introduce this Supplement, and then we ſhall have the accurate value of a, inſtead of an approximation to that value. Here the firſt Sign is to be taken if n is an odd number, and the other when it is even. Thus if $n=1$, and conſequently $k=a$, and $l=b$, we ſhall have $\frac{bp+aq}{q}-\frac{bp}{q}=a$. Or if $n=2$, and $l=c$, then $\frac{bp+aq}{q}-\frac{cp+bq}{q}\times\frac{p}{q}+\frac{cp^2}{q^2}=a$. Or if $n=3$, $l=d$, then $\frac{bp+aq}{q}-\frac{cp+bq}{q}\times\frac{p}{q}+\frac{dp+cq}{q}\times\frac{p^2}{q^2}-\frac{dp^3}{q^3}=a$. And ſo on. Here the taking in of the Supplement always compleats the value of a, and makes it perfect, whether the Series be converging or diverging; which will always be the beſt way of proceeding, when that Supplement can readily be known. But as this rarely happens, in ſuch infinite Series as generally occur, we muſt have recourſe to infinite converging Series, wherein this Supplement, as well as the Terms of the Series, are infinitely diminiſh'd; and therefore after a competent number of them are collected, the reſt may be all neglected *in infinitum.*

From this general Series, the better to aſſiſt the Imagination, we will deſcend to a few particular Inſtances of converging Series in pure Numbers. Let the Coefficients a, b, c, d, &c. be expounded by 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. reſpectively; then $\frac{\frac{1}{2}p+q}{q}-\frac{\frac{1}{3}p+\frac{1}{2}q}{q}\times\frac{p}{q}+\frac{\frac{1}{4}p+\frac{1}{3}q}{q}\times\frac{p^2}{4^2}$, &c. $=1$, or $\frac{p+2q}{2q}-\frac{2p+3q}{2\times 3q}\times\frac{p}{q}+\frac{3p+4q}{3\times 4q}\times\frac{p^2}{q^2}-\frac{4p+5q}{4\times 5q}\times\frac{p^3}{q^3}$, &c. $=1$. That the Series hence ariſing may converge, make p leſs

than q in any given ratio, ſuppoſe $\frac{p}{q} = \frac{1}{2}$, or $p = 1$, $q = 2$, then $\frac{5}{4} - \frac{1}{5} \times \frac{1}{2} + \frac{11}{24} \times \frac{1}{4} - \frac{7}{20} \times \frac{1}{8}$, &c. $= 1$. That is, this Series of Fractions, which is computed by Binary Arithmetick, or by the Reciprocals of the Powers of Two, if infinitely continued will finally be equal to Unity. Or if we deſire to ſtop at theſe four Terms, and inſtead of the reſt *ad infinitum* if we would introduce the Supplement which is equivalent to them, and which is here known to be $\frac{1}{5} \times \frac{1}{16}$, or $\frac{1}{80}$, we ſhall have $\frac{5}{4} - \frac{1}{8} + \frac{11}{96} - \frac{7}{160} + \frac{1}{80} = 1$, as is eaſy to prove. Or let the ſame Coefficients be expounded by $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}$, &c. then it will be $\frac{2q-p}{2q} + \frac{3q-2p}{2 \times 3q} \times \frac{p}{q} + \frac{4q-3p}{3 \times 4q} \times \frac{p^2}{q^2} + \frac{5q-4p}{4 \times 5q} \times \frac{p^3}{q^3}$, &c. $= 1$. This Series may either be continued infinitely, or may be ſum'd after any number of Terms expreſs'd by n, by introducing the Supplement $\frac{+p^n}{\overline{n+1} \times q^n}$ inſtead of all the reſt. Or more particularly, if we make $q = 5p$, then $\frac{9}{2 \times 5} + \frac{13}{6 \times 5^2} + \frac{17}{12 \times 5^3} + \frac{21}{20 \times 5^4} + \frac{25}{30 \times 5^5}$, &c. $= 1$, which is a Number expreſs'd by Quinary Arithmetick. And this is eaſily reduced to the Decimal Scale, by writing $\frac{2}{10}$ for $\frac{1}{5}$, and reducing the Coefficients; for then it will become 0,99999, &c. $= 1$. Now if we take theſe five Terms, together with the Supplement, we ſhall have exactly $\frac{9}{2 \times 5} + \frac{13}{6 \times 5^2} + \frac{17}{12 \times 5^3} + \frac{21}{20 \times 5^4} + \frac{25}{30 \times 5^5} + \frac{1}{6 \times 5^5} = 1$. Again, if we make here $3q = 100p$, we ſhall have the Series $\frac{200-3}{1 \times 2} \times \frac{1}{100} + \frac{300-6}{2 \times 3} \times \frac{3}{10000} + \frac{400-9}{3 \times 4} \times \frac{9}{1000000} + \frac{500-12}{4 \times 5} \times \frac{27}{100000000}$, &c. $= 1$, which converges very faſt. And if we would reduce this to the regular Decimal Scale of Arithmetick, (which is always ſuppos'd to be done, before any particular Problem can be ſaid to be compleatly ſolved,) we muſt ſet the Terms, when decimally reduced, orderly under one another, that their Amount or Aggregate may be diſcover'd; and then they will ſtand as in the Margin. Here the Aggregate of the firſt five Terms is 0,99999999595, which is a near Approximation to the Amount of the whole infinite Series, or to Unity. And if, for proof-ſake, we add to this the Supplement $\frac{+p^n}{\overline{n+1}\, q^n} = \frac{p^5}{6q^5} = 0{,}00000000405$, the whole will be Unity exactly.

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0,985
   147
     29325
       6588
        15795
0,99999999595
          405
1,00000000000
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There are alſo other Methods of forming converging Series, whether general or particular, which ſhall approximate to a known quantity, and therefore will be very proper to explain the nature of Convergency, and to ſhew how the Supplement is to be introduced, when it can be done, in order to make the Series finite; which of late has been call'd the Summing of a Series. Let A, B, C, D, E, *&c.* and a, b, c, d, e, &c. be any two Progreſſions of Terms, of which A is to be expreſs'd by a Series, either finite or infinite, compos'd of itſelf and the other Terms. Suppoſe therefore the firſt Term of the Series to be a, and that p is the ſupplement to the value of a. Then is $A = a + p$, or $p = \frac{A-a}{1}$. As this is the whole Supplement, in order to form a Series, I ſhall only take ſuch a part of it as is denominated by the Fraction $\frac{b}{B}$, and put q for the ſecond Supplement. That is, I will aſſume $\frac{A-a}{1} = (p =) \frac{A-a}{1} \times \frac{b}{B} + q$; or $q = (\frac{A-a}{1} \times \overline{1 - \frac{b}{B}} =) \frac{A-a}{B} \times \frac{B-b}{1}$. Again, as this is the whole value of the Supplement q, I ſhall only aſſume ſuch a part of it as is denominated by the Fraction $\frac{c}{C}$, and for the next Supplement put r. That is, $\frac{A-a}{B} \times \frac{B-b}{1} = (q =) \frac{A-a}{B} \times \frac{B-b}{C} c + r$, or $r = (\frac{A-a}{B} \times \frac{B-b}{1} \times 1 - \frac{c}{C} =) \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{1}$. Now as this is the whole value of the Supplement r, I only aſſume ſuch a part of it as is denominated by the Fraction $\frac{d}{D}$, and for the next Supplement put s. That is, $\frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{1} = (r =) \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} d + s$, or $s = \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{1} \times \overline{1 - \frac{d}{D}} = \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} \times \frac{D-d}{1}$. And ſo on as far as we pleaſe. So that at laſt we have the value of $A = a + p$, where the Supplement $p = \frac{A-a}{B} b + q$, where the ſecond Supplement $q = \frac{A-a}{B} \times \frac{B-b}{C} c + r$, where $r = \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} d + s$, where $s = \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} \times \frac{D-d}{E} e + t$. And ſo on *ad infinitum.* That is finally $A = a + \frac{A-a}{B} b + \frac{A-a}{B} \times \frac{B-b}{C} c + \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} d + \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} \times \frac{D-d}{E} e$, &c. where A, B, C, D, E, *&c.* and a, b, c, d, e, &c. may be any two Progreſſions of Numbers whatever, whether regular or deſultory, aſcending or deſcending. And when

it

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it happens in theſe Progreſſions, that either $A = a$, or $B = b$, or $C = c$, &c. then the Series terminates of itſelf, and exhibits the value of A in a finite number of Terms: But in other caſes it approximates indefinitely to the value of A. But in the caſe of an infinite Approximation, the ſaid Progreſſions ought to proceed regularly, according to ſome ſtated Law. Here it will be eaſy to obſerve, that if K and k are put to repreſent any two Terms indefinitely in the aforeſaid Progreſſions, whoſe places are denoted by the number n, and if L and l are the Terms immediately following; then the Term in the Series denoted by $n + 1$ will be form'd from the preceding Term, by multiplying it by $\frac{K - k}{kL}l$. As if $n = 1$, then $K = A$, $k = a$, $L = B$, $l = b$, and the ſecond Term will be $a + \frac{A - a}{aB}b = \frac{A - a}{B}b$. If $n = 2$, then $K = B$, $k = b$, $L = C$, $l = c$, and the third Term will be $\frac{A-a}{B}b \times \frac{B-b}{bC}c = \frac{A-a}{B} \times \frac{B-b}{C}c$; and ſo of the reſt. And whenever it ſhall happen that $L = l$, then the Series will ſtop at this Term, and proceed no farther. And the Series approximates ſo much the faſter, *cæteris paribus*, as the Numbers A, B, C, D, &c. and a, b, c, d, &c. approach nearer to each other reſpectively.

Now to give ſome Examples in pure Numbers. Let A, B, C, D, &c. $= 2, 2, 2, 2$, &c. and a, b, c, d, &c. $= 1, 1, 1, 1$, &c. then we ſhall have $2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$, &c. And ſo always, when the given Progreſſions are Ranks of equals, the Series will be a Geometrical Progreſſion. If we would have this Progreſſion ſtop at the next Term, we may either ſuppoſe the firſt given Progreſſion to be 2, 2, 2, 2, 2, 1, or the ſecond to be 1, 1, 1, 1, 1, 2, 'tis all one which. For in either caſe we ſhall have $L = l$, that is $F = f$, and therefore the laſt Term muſt be multiply'd by $\frac{K-k}{k}$, or $\frac{F-e}{e} = 1$. Then the Progreſſion or Series becomes $2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}$. Again, if A, B, C, D, &c. $= 5, 5, 5, 5$, &c. and a, b, c, d, &c. $= 4, 4, 4, 4$, &c. then $5 = 4 + \frac{4}{5} + \frac{4}{25} + \frac{4}{125} + \frac{4}{625}$, &c. or $\frac{1}{4} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625}$, &c. Or if A, B, C, D, &c. $= 4, 4, 4, 4$, &c. and a, b, c, d, &c. $= 5, 5, 5, 5$, &c. then $4 = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \frac{5}{256}$, &c. If A, B, C, D, &c. $= 5, 5, 5, 5$, &c. and a, b, c, d, &c. $= 6, 7, 8, 9$, &c. then $5 = 6 - \frac{1}{5}7 + \frac{1}{5} \times \frac{2}{5}8 - \frac{1}{5} \times \frac{2}{5} \times \frac{3}{5}9 + \frac{1}{5} \times \frac{2}{5} \times \frac{3}{5} \times \frac{4}{5}10$, &c. If we would have the Series ſtop here, or if we would find one more Term, or Supplement, which ſhould be equivalent to all the reſt *ad infinitum*, (which indeed might be deſirable here, and in ſuch caſes as this, becauſe of the ſlow Convergency, or rather Divergency of the Series,) ſuppoſe $F = f$, and therefore $\frac{E-e}{e} = \frac{5-10}{10} = -\frac{1}{2}$ muſt be multiply'd by the laſt Term. So that the Series becomes $5 = 6 - \frac{1}{5}7 + \frac{1}{5} \times \frac{2}{5}8 - \frac{1}{5} \times \frac{2}{5} \times \frac{3}{5}9 + \frac{1}{5} \times \frac{2}{5} \times \frac{3}{5} \times \frac{4}{5}10 - \frac{1}{5} \times \frac{2}{5} \times \frac{3}{5} \times \frac{4}{5}5$. If A, B, C, D, &c. $=$ 2, 3, 4, 5, &c. and a, b, c, d, &c. $=$ 1, 2, 3, 4, &c. then $2 = 1 + \frac{1}{3}2 + \frac{1}{3} \times \frac{1}{4}3 + \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5}4 + \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} \times \frac{1}{6}5$, &c. If A, B, C, D, &c. $=$ 1, 2, 3, 4, &c. and a, b, c, d, &c. $=$ 2, 3, 4, 5, &c. then $1 = 2 - \frac{1}{2}3 + \frac{1}{2} \times \frac{1}{3}4 - \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}5 + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5}6$, &c. And from this general Series may infinite other particular Series be eaſily derived, which ſhall perpetually converge to given Quantities; the chief uſe of which Speculation, I think, will be, to ſhew us the nature of Convergency in general.

There are many other ſuch like general Series that may be readily form'd, which ſhall converge to a given Number. As if I would conſtruct a Series that ſhall converge to Unity, I ſet down 1, together with a Rank of Fractions, both negative and affirmative, as here follows.

$$\begin{array}{l}
1 - \frac{a}{A} - \frac{b}{B} - \frac{c}{C} - \frac{d}{D} - \frac{e}{E}, \text{\&c.} \\
\quad + \frac{a}{A} + \frac{b}{B} + \frac{c}{C} + \frac{d}{D} + \frac{e}{E}, \text{\&c.} \\
\hline
\frac{A+a}{A} + \frac{Ab-Ba}{AB} + \frac{Bc-Cb}{BC} + \frac{Cd-Dc}{CD} + \frac{De-Ed}{DE}, \text{\&c.} = 1.
\end{array}$$

Then proceeding obliquely, I collect the Terms of each Series together, by adding the two firſt, then the two ſecond, and ſo on. So that the whole Series thus conſtructed muſt neceſſarily be equal to Unity; which alſo is manifeſt by a bare Inſpection of the Series. From this Series it is eaſy to deſcend to any number of particular Caſes. As if we make A, B, C, D, &c. $=$ 2, 3, 4, 5, &c. and a, b, c, d, &c. $=$ 1, 1, 1, 1, &c. then $\frac{3}{2} - \frac{1}{2 \times 3} - \frac{1}{3 \times 4} - \frac{1}{4 \times 5} - \frac{1}{5 \times 6}$, &c. $= 1$. Or $\frac{1}{2} = \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \frac{1}{5 \times 6}$, &c. And ſo in all other Caſes. The Series will ſtop at a finite number of Terms, whenſoever you omit to take in the firſt part of the Numerator of any Term. As here $\frac{3}{2} - \frac{1}{2 \times 3} - \frac{1}{3 \times 4} - \frac{1}{4 \times 5} - \frac{1}{5 \times 6} - \frac{1}{6} = 1$.

Laſtly, to conſtruct one more Series of this kind, which ſhall converge to Unity; I ſet down 1, with a Rank of Fractions along with it, both affirmative and negative, ſuch as are ſeen here below; which being added together obliquely as before, will produce the following Series.

$$\begin{array}{l}
1 + \frac{a}{A} + \frac{ab}{AB} + \frac{abc}{ABC} + \frac{abcd}{ABCD} + \frac{abcde}{ABCDE},\ \&c. \\
\quad - \frac{a}{A} - \frac{ab}{AB} - \frac{abc}{ABC} - \frac{abcd}{ABCD} - \frac{abcde}{ABCDE},\ \&c. \\
\hline
\frac{A-a}{A} + \frac{B-b}{AB}a + \frac{C-c}{ABC}ab + \frac{D-d}{ABCD}abc + \frac{E-e}{ABCDE}abcd,\ \&c. = 1.
\end{array}$$

This Series may be made to ſtop at any finite number of Terms, if you omit to take in the latter part of the Binomial in any Term. Or you may derive particular Series from it, which ſhall have any Rate of Convergency.

For an Example of this Series, make A, B, C, D, &c. $=$ 3, 3, 3, 3, &c. and a, b, c, d, &c. $=$ 1, 1, 1, 1, &c. then $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81}$, &c. $= 1$, or $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$, &c. $= \frac{1}{2}$. And whenever A, B, C, &c. and a, b, c, &c. are Ranks of Equals, the Series will be a Geometrical Progreſſion.

Again, make A, B, C, D, &c. $=$ 2, 3, 4, 5, &c. and a, b, c, d, &c. $=$ 1, 1, 1, 1, &c. then $\frac{1}{2} + \frac{2}{2 \times 3} + \frac{3}{2 \times 3 \times 4} + \frac{4}{2 \times 3 \times 4 \times 5} + \frac{5}{2 \times 3 \times 4 \times 5 \times 6}$, &c. $= 1$. Or in a finite number of Terms $\frac{1}{2} + \frac{1}{3} + \frac{1}{2 \times 4} + \frac{1}{2 \times 3 \times 5} + \frac{1}{2 \times 3 \times 4 \times 5} = 1$. And the like may be obſerved of others in an infinite variety.

And thus having prepared the way for what follows, by explaining the nature of infinite Series in general, by diſcovering their origin and manner of convergency, and by ſhewing their connexion with our common Arithmetick; I ſhall now return to our Author's Methods of Operation, or to the Reduction of compound Quantities to ſuch infinite Series.

SECT. II. *The Reſolution of ſimple Equations, or pure Powers, by Infinite Series.*

3, 4. THE Author begins his Reduction of compound Quantities, to an equivalent infinite Series of ſimple Terms, firſt by ſhewing how the Proceſs may be perform'd in Diviſion. Now in his Example the manner of the Operation is thus, in imitation

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tation of the uſual praxis of Diviſion in Numbers. In order to obtain the Quotient of aa divided by $b+x$, or to reſolve the compound Fraction $\frac{aa}{b+x}$ into a Series of ſimple Terms, firſt find the Quotient of aa divided by b, the firſt Term of the Diviſor. This is $\frac{aa}{b}$, which write in the Quote. Then multiply the Diviſor by this Term, and ſet the Product $aa+\frac{aax}{b}$ under the Dividend, from whence it muſt be ſubtracted, and will leave the Remainder $-\frac{aax}{b}$. Then to find the next Term (or Figure) of the Quotient, divide the Remainder by the firſt Term of the Diviſor, or by b, and put the Quotient $-\frac{aax}{b^2}$ for the ſecond Term of the Quote. Multiply the Diviſor by this ſecond Term, and the Product $-\frac{aax}{b}-\frac{aaxx}{bb}$ ſet orderly under the laſt Remainder; from whence it muſt be ſubtracted, to find the new Remainder $+\frac{aaxx}{bb}$. Then to find the next Term of the Quotient, you are to proceed with this new Remainder as with the former; and ſo on *in infinitum*. The Quotient therefore is $\frac{a^2}{b}-\frac{a^2x}{b^2}+\frac{a^2x^2}{b^3}-\frac{a^2x^3}{b^4}$, &c. (or $\frac{a^2}{b}$ into $1-\frac{x}{b}+\frac{x^2}{b^2}-\frac{x^3}{b^3}$, &c.) So that by this Operation the Number or Quantity $\frac{aa}{b+x}$, (or $a^2\times\overline{b+x}|^{-1}$) is reduced from that Scale in Arithmetick whoſe Root is $b+x$, to an equivalent Number, the Root of whoſe Scale, (or whoſe converging quantity) is $\frac{x}{b}$. And this Number, or infinite Series thus found, will converge ſo much the faſter to the truth, as b is greater than x.

To apply this, by way of illuſtration, to an inſtance or two in common Numbers. Suppoſe we had the Fraction $\frac{1}{7}$, and would reduce it from the ſeptenary Scale, in which it now appears, to an equivalent Series, that ſhall converge by the Powers of 6. Then we ſhall have $\frac{1}{7}=\frac{1}{6+1}$; and therefore in the foregoing general Fraction $\frac{aa}{b+x}$, make $a=1$, $b=6$, and $x=1$, and the Series will become $\frac{1}{6}-\frac{1}{6^2}+\frac{1}{6^3}-\frac{1}{6^4}$, &c. which will be equivalent to $\frac{1}{7}$. Or if we would reduce it to a Series converging by the Powers of 8, becauſe $\frac{1}{7}=\frac{1}{8-1}$, make $a=1$, $b=8$, and $x=-1$, then $\frac{1}{7} = \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + \frac{1}{8^4}$, &c. which Series will converge faſter than the former. Or if we would reduce it to the common Denary (or Decimal) Scale, becauſe $\frac{1}{7} = \frac{1}{10 - 3}$, make $a = 1$, $b = 10$, and $x = -3$; then $\frac{1}{7} = \frac{1}{10} + \frac{3}{100} + \frac{9}{1000} + \frac{27}{10000} + \frac{81}{100000}$, &c. $= 0,1428$, &c. as may be eaſily collected. And hence we may obſerve, that this or any other Fraction may be reduced a great variety of ways to infinite Series; but that Series will converge faſteſt to the truth, in which b ſhall be greateſt in reſpect of x. But that Series will be moſt eaſily reduced to the common Arithmetick, which converges by the Powers of 10, or its Multiples. If we ſhould here reſolve 7 into the parts $3 + 4$, or $2 + 5$, or $1 + 6$, *&c.* inſtead of converging we ſhould have diverging Series, or ſuch as require a Supplement to be taken in.

And we may here farther obſerve, that as in Diviſion of common Numbers, we may ſtop the proceſs of Diviſion whenever we pleaſe, and inſtead of all the reſt of the Figures (or Terms) *ad infinitum*, we may write the Remainder as a Numerator, and the Diviſor as the Denominator of a Fraction, which Fraction will be the Supplement to the Quotient: ſo the ſame will obtain in the Diviſion of Species. Thus in the preſent Example, if we will ſtop at the firſt Term of the Quotient, we ſhall have $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{b \times \overline{b+x}}$. Or if we will ſtop at the ſecond Term, then $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{b^2} + \frac{a^2x^2}{b^3+b^2x}$. Or if we will ſtop at the third Term, then $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{b^2} + \frac{aax^2}{b^3} - \frac{a^2x^3}{b^4+b^3x}$. And ſo in the ſucceeding Terms, in which theſe Supplements may always be introduced, to make the Quotient compleat. This Obſervation will be found of good uſe in ſome of the following Speculations, when a complicate Fraction is not to be intirely reſolved, but only to be depreſs'd, or to be reduced to a ſimpler and more commodious form.

Or we may hence change Diviſion into Multiplication. For having found the firſt Term of the Quotient, and its Supplement, or the Equation $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{b^2+bx}$; multiplying it by $\frac{x}{b}$, we ſhall have $\frac{aax}{b^2+bx} = \frac{aax}{b^2} - \frac{a^2x^2}{b^3+b^2x}$, ſo that ſubſtituting this value of $\frac{aax}{b^2+bx}$ in the firſt Equation, it will become $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{b^2} + \frac{a^2x^2}{b^3+b^2x}$, where the two firſt Terms of the Quotient are now known.

Multiply this by $\frac{x^2}{b^2}$, and it will become $\frac{a^2x^2}{b^3+b^2x} = \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} + \frac{a^2x^4}{b^5+b^4x}$, which being ſubſtituted in the laſt Equation, it will become $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} + \frac{a^2x^4}{b^5+b^4x}$, where the four firſt Terms of the Quotient are now known. Again, multiply this Equation by $\frac{x^4}{b^4}$, and it will become $\frac{aax^4}{b^5+b^4x} = \frac{a^2x^4}{b^5} - \frac{a^2x^5}{b^6} + \frac{a^2x^6}{b^7} - \frac{a^2x^7}{b^8} + \frac{a^2x^8}{b^9+b^8x}$, which being ſubſtituted in the laſt Equation, it will become $\frac{aa}{b+x} = \frac{a^2}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} + \frac{a^2x^4}{b^5} - \frac{a^2x^5}{b^6} + \frac{a^2x^6}{b^7} - \frac{a^2x^7}{b^8} + \frac{a^2x^8}{b^9+b^8x}$, where eight of the firſt Terms are now known. And ſo every ſucceeding Operation will double the number of Terms, that were before found in the Quotient.

This method of Reduction may be thus very conveniently imitated in Numbers, or we may thus change Diviſion into Multiplication. Suppoſe (for inſtance) I would find the Reciprocal of the Prime Number 29, or the value of the Fraction $\frac{1}{29}$ in Decimal Numbers. I divide 1,0000, *&c.* by 29, in the common way, ſo far as to find two or three of the firſt Figures, or till the Remainder becomes a ſingle Figure, and then I aſſume the Supplement to compleat the Quotient. Thus I ſhall have $\frac{1}{29} = 0{,}03448\frac{8}{29}$ for the compleat Quotient, which Equation if I multiply by the Numerator 8, it will give $\frac{8}{29} = 0{,}27584\frac{64}{29}$, or rather $\frac{8}{29} = 0{,}27586\frac{6}{29}$. I ſubſtitute this inſtead of the Fraction in the firſt Equation, and I ſhall have $\frac{1}{29} = 0{,}0344827586\frac{6}{29}$. Again, I multiply this Equation by 6, and it will give $\frac{6}{29} = 0{,}2068965517\frac{7}{29}$, and then by Subſtitution $\frac{1}{29} = 0{,}03448275862068965517\frac{7}{29}$. Again, I multiply this Equation by 7, and it becomes $\frac{7}{29} = 0{,}24137931034482758620\frac{20}{29}$, and then by Subſtitution $\frac{1}{29} = 0{,}0344827586206896551724137931034482758620\frac{20}{29}$, where every Operation will at leaſt double the number of Figures found by the preceding Operation. And this will be an eaſy Expedient for converting Diviſion into Multiplication in all Caſes. For the Reciprocal of the Diviſor being thus found, it may be multiply'd into the Dividend to produce the Quotient.

Now as it is here found, that $\frac{aa}{b+x} = \frac{aa}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4}$, &c. which Series will converge when b is greater than x; ſo when it happens to be otherwiſe, or when x is greater than b, that the Powers of x may be in the Denominators we muſt have recourſe to the other Caſe of Diviſion, in which we ſhall find $\frac{aa}{x+b} = \frac{aa}{x} - \frac{a^2b}{x^2} + \frac{a^2b^2}{x^3} - \frac{a^2b^3}{x^4}$, &c. and where the Diviſion is perform'd as before.

5, 6. In theſe Examples of our Author, the Proceſs of Diviſion (for the exerciſe of the Learner) may be thus exhibited :

$$\begin{array}{l}
1+x^2)\,1+0\,(1-x^2+x^4-x^6+x^8, \&c. \\
\quad\;\; \underline{1+x^2} \\
\quad\;\; 0-x^2+0 \\
\quad\quad\; \underline{-x^2-x^4} \\
\quad\quad\quad 0+x^4+0 \\
\quad\quad\quad\;\; \underline{+x^4+x^6} \\
\quad\quad\quad\quad 0-x^6+0 \\
\quad\quad\quad\quad\;\; \underline{-x^6-x^8} \\
\quad\quad\quad\quad\quad 0+x^8
\end{array}
\qquad
\begin{array}{l}
1+x^{\frac{1}{2}}-3x)\,2x^{\frac{1}{2}} * -x^{\frac{3}{2}}\,(2x^{\frac{1}{2}}-2x+7x^{\frac{3}{2}}-13x^2, \&c. \\
\quad\quad\quad\quad\;\; \underline{2x^{\frac{1}{2}}+2x-6x^{\frac{3}{2}}} \\
\quad\quad\quad\quad\quad -2x+5x^{\frac{3}{2}} \\
\quad\quad\quad\quad\quad \underline{-2x-2x^{\frac{3}{2}}+6x^2} \\
\quad\quad\quad\quad\quad\quad +7x^{\frac{3}{2}}-6x^2 \\
\quad\quad\quad\quad\quad\quad \underline{+7x^{\frac{3}{2}}+7x^2-21x^{\frac{5}{2}}} \\
\quad\quad\quad\quad\quad\quad\quad -13x^2+21x^{\frac{5}{2}}
\end{array}$$

Now in order to a due Convergency, in each of theſe Examples, we muſt ſuppoſe x to be leſs than Unity; and if x be greater than Unity, we muſt invert the Terms, and then we ſhall have $\frac{1}{xx+1} = \frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6} - \frac{1}{x^8}$, &c. and $\frac{-x^{\frac{3}{2}}+2x^{\frac{1}{2}}}{-3x+x^{\frac{1}{2}}+1} = \frac{1}{3}x^{\frac{1}{2}} + \frac{1}{9} - \frac{14}{27x^{\frac{1}{2}}} - \frac{11}{81x}$, &c.

7, 8, 9, 10. This Notation of Powers and Roots by integral and fractional, affirmative and negative, general and particular Indices, was certainly a very happy Thought, and an admirable Improvement of Analyticks, by which the practice is render'd eaſy, regular, and univerſal. It was chiefly owing to our Author, at leaſt he carried on the Analogy, and made it more general. A Learner ſhould be well acquainted with this Notation, and the Rules of its ſeveral Operations ſhould be very familiar to him, or otherwiſe he will often find himſelf involved in difficulties. I ſhall not enter into any farther diſcuſſion of it here, as not properly belonging to this place, or ſubject, but rather to the vulgar Algebra.

11. The Author proceeds to the Extraction of the Roots of pure Equations, which he thus performs, in imitation of the uſual Proceſs in Numbers. To extract the Square-root of $aa+xx$; firſt the Root of aa is a, which muſt be put in the Quote. Then the Square of this, or aa, being ſubtracted from the given Power, leaves $+xx$ for a Reſolvend. Divide this by twice the Root, or $2a$, which is

the firſt part of the Diviſor, and the Quotient $\frac{xx}{2a}$ muſt be made the ſecond Term of the Root, as alſo the ſecond Term of the Diviſor. Multiply the Diviſor thus compleated, or $2a+\frac{xx}{2a}$, by the ſecond Term of the Root, and the Product $xx+\frac{x^4}{4aa}$ muſt be ſubtracted from the Reſolvend. This will leave $-\frac{x^4}{4a^2}$ for a new Reſolvend, which being divided by the firſt Term of the double Root, or $2a$, will give $-\frac{x^4}{8a^3}$ for the third Term of the Root. Twice the Root before found, with this Term added to it, or $2a+\frac{x^2}{a}-\frac{x^4}{8a^3}$, being multiply'd by this Term, the Product $-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}$ muſt be ſubtracted from the laſt Reſolvend, and the Remainder $+\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$ will be a new Reſolvend, to be proceeded with as before, for finding the next Term of the Root; and ſo on as far as you pleaſe. So that we ſhall have $\sqrt{aa+xx}=a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}$, &c.

It is eaſy to obſerve from hence, that in the Operation every new Column will give a new Term in the Quote or Root; and therefore no more Columns need be form'd than it is intended there ſhall be Terms in the Root. Or when any number of Terms are thus extracted, as many more may be found by Diviſion only. Thus having found the three firſt Terms of the Root, $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}$, by their double $2a+\frac{x^2}{a}-\frac{x^4}{4a^3}$, dividing the third Remainder or Reſolvend $+\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$, the three firſt Terms of the Quotient $\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\frac{7x^{10}}{256a^9}$ will be the three ſucceeding Terms of the Root.

The Series $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}$, &c. thus found for the ſquare-root of the irrational quantity $aa+xx$, is to be underſtood in the following manner. In order to a due convergency a is to be ſuppos'd greater than x, that the Root or converging quantity $\frac{x}{a}$ may be leſs than Unity, and that a may be a near approximation to the ſquare-root required. But as this is too little, it is encreaſed by the ſmall quantity $\frac{x^2}{2a}$, which now makes it too big. Then by the next

Operation it is diminiſh'd by the ſtill ſmaller quantity $\frac{x^4}{8a^3}$; which diminution being too much, it is again encreas'd by the very ſmall quantity $\frac{x^6}{16a^5}$, which makes it too great, in order to be farther diminiſh'd by the next Term. And thus it proceeds *in infinitum*, the Augmentations and Diminutions continually correcting one another, till at laſt they become inconſiderable, and till the Series (ſo far continued) is a ſufficiently near Approximation to the Root required.

12. When a is leſs than x, the order of the Terms muſt be inverted, or the ſquare-root of $xx + aa$ muſt be extracted as before; in which caſe it will be $x + \frac{aa}{2x} - \frac{a^4}{8x^3}$, &c. And in this Series the converging quantity, or the Root of the Scale, will be $\frac{a}{x}$. Theſe two Series are by no means to be underſtood as the two different Roots of the quantity $aa + xx$; for each of the two Series will exhibit thoſe two Roots, by only changing the Signs. But they are accommodated to the two Caſes of Convergency, according as a or x may happen to be the greater quantity.

I ſhall here reſolve the foregoing Quantity after another manner, the better to prepare the way for what is to follow. Suppoſe then $yy = aa + xx$, where we may find the value of the Root y by the following Proceſs: $yy = aa + xx =$ (if $y = a + p$) $aa + 2ap + pp$; or $2ap + pp = xx =$ (if $p = \frac{xx}{2a} + q$) $xx + 2aq + \frac{x^4}{4a^2} + \frac{x^2q}{a} + qq$; or $2aq + \frac{x^2q}{a} + qq = -\frac{x^4}{4a^2} =$ (if $q = -\frac{x^4}{8a^3} + r$) $-\frac{x^4}{4a^2} + 2ar - \frac{x^6}{8a^4} + \frac{x^2r}{a} + \frac{x^8}{64a^6} - \frac{x^4r}{4a^3} + r^2$; or $2ar + \frac{xxr}{a} - \frac{x^4r}{4a^3} + rr = \frac{x^6}{8a^4} - \frac{x^8}{64a^6} =$ (if $r = \frac{x^6}{10a^5} + s$) &c. which Proceſs may be thus explain'd in words.

In order to find $\sqrt{aa + xx}$, or the Root y of this Equation $yy = aa + xx$, ſuppoſe $y = a + p$, where a is to be underſtood as a pretty near Approximation to the value of y, (the nearer the better,) and p is the ſmall Supplement to that, or the quantity which makes it compleat. Then by Subſtitution is derived the firſt Supplemental Equation $2ap + pp = xx$, whoſe Root p is to be found. Now as $2ap$ is much bigger than pp, (for $2a$ is bigger than the Supplement p,) we ſhall have nearly $p = \frac{xx}{2a}$, or at leaſt we ſhall have exactly $p = \frac{xx}{2a} + q$, ſuppoſing q to repreſent the ſecond Supplement of the Root. Then by Subſtitution $2aq + \frac{xx}{a}q + qq = -\frac{x^4}{4a^2}$ will be the ſecond Supplemental Equation, whoſe Root q is the ſecond Supplement. Therefore $\frac{xx}{a}q$ will be a little quantity, and qq much leſs, ſo that we ſhall have nearly $q = -\frac{x^4}{8a^3}$, or accurately $q = -\frac{x^4}{8a^3} + r$, if r be made the third Supplement to the Root. And therefore $2ar + \frac{xx}{a}r - \frac{x^4}{4a^3}r + r^2 = \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$ will be the third Supplemental Equation, whoſe Root is r. And thus we may go on as far as we pleaſe, to form Reſidual or Supplemental Equations, whoſe Roots will continually grow leſs and leſs, and therefore will make nearer and nearer Approaches to the Root y, to which they always converge. For $y = a + p$, where p is the Root of this Equation $2ap + pp = xx$. Or $y = a + \frac{xx}{2a} + q$, where q is the Root of this Equation $2aq + \frac{xx}{a}q + qq = -\frac{x^4}{4a^2}$. Or $y = a + \frac{xx}{2a} - \frac{x^4}{8a^3} + r$, where r is the Root of this Equation $2ar + \frac{xx}{a}r - \frac{x^4}{4a^3}rr = \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$. And ſo on. The Reſolution of any one of theſe Quadratick Equations, in the ordinary way, will give the reſpective Supplement, which will compleat the value of y.

I took notice before, upon the Article of Diviſion, of what may be call'd a Compariſon of Quotients; or that one Quotient may be exhibited by the help of another, together with a Series of known or ſimple Terms. Here we have an Inſtance of a like Compariſon of Roots; or that the Root of one Equation may be expreſs'd by the Root of another, together with a Series of known or ſimple Terms, which will hold good in all Equations whatever. And to carry on the Analogy, we ſhall hereafter find a like Compariſon of Fluents; where one Fluent, (ſuppoſe, for inſtance, a Curvilinear Area,) will be expreſs'd by another Fluent, together with a Series of ſimple Terms. This I thought fit to inſinuate here, by way of anticipation, that I might ſhew the conſtant uniformity and harmony of Nature, in theſe Speculations, when they are duly and regularly purſued.

But I ſhall here give, *ex abundanti*, another Method for this, and ſuch kind of Extractions, tho' perhaps it may more properly belong to the Reſolution of Affected Equations, which is ſoon to follow; however it may ſerve as an Introduction to their Solution.

The firſt Reſidual or Supplemental Equation in the foregoing Proceſs was $2ap + pp = xx$, which may be reſolved in this manner. Becauſe $p = \frac{xx}{2a+p}$, it will be by Diviſion $p = \frac{xx}{2a} - \frac{x^2p}{4a^2} + \frac{x^2p^2}{8a^3} - \frac{x^2p^3}{16a^4} + \frac{x^2p^4}{32a^5}$, &c. Divide all the Terms of this Series (except the firſt) by p, and then multiply them by the whole Series, or by the value of p, and you will have $p = \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^4p}{8a^4} - \frac{3x^4p^2}{32a^5} + \frac{x^4p^3}{16a^6}$, &c. where the two firſt Terms are clear'd of p. Divide all the Terms of this Series, except the two firſt, by p, and multiply them by the value of p, or by the firſt Series, and you will have a Series for p in which the three firſt Terms are clear'd of p. And by repeating the Operation, you may clear as many Terms of p as you pleaſe. So that at laſt you will have $p = \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}$, &c. which will give the ſame value of y as before.

13, 14, 15, 16, 17, 18. The ſeveral Roots of theſe Examples, and of all other pure Powers, whether they are Binomials, Trinomials, or any other Multinomials, may be extracted by purſuing the Method of the foregoing Proceſs, or by imitating the like *Praxes* in Numbers. But they may be perform'd much more readily by general Theorems computed for that purpoſe. And as there will be frequent occaſion, in the enſuing Treatiſe, for certain general Operations to be perform'd with infinite Series, ſuch as Multiplication, Diviſion, raiſing of Powers, and extracting of Roots; I ſhall here derive ſome Theorems for thoſe purpoſes.

I. Let $A+B+C+D+E$, &c. $P+Q+R+S+T$, &c. and $\alpha+\beta+\gamma+\delta+\varepsilon$, &c. repreſent the Terms of three ſeveral Series reſpectively, and let $A+B+C+D+E$, &c. into $P+Q+R+S+T$, &c. $= \alpha+\beta+\gamma+\delta+\varepsilon$, &c. Then by the known Rules of Multiplication, by which every Term of one Factor is to be multiply'd into every Term of the other, it will be $\alpha = AP$, $\beta = AQ + BP$, $\gamma = AR + BQ + CP$, $\delta = AS + BR + CQ + DP$, $\varepsilon = AT + BS + CR + DQ + EP$; and ſo on. Then by Subſtitution it will be

$$\begin{array}{l} \overline{A+B+C+D+E, \&c.} \times \overline{P+Q+R+S+T, \&c.} = AP + BP + CP + DP + EP, \&c. \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad + AQ + BQ + CQ + DQ \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad + AR + BR + CR \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad + AS + BS \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + AT \end{array}$$

And this will be a ready Theorem for the Multiplication of any infinite Series into each other; as in the following Example.

$$\overset{(A)}{a}+\overset{(B)}{\tfrac{1}{2}x}+\overset{(C)}{\frac{x^2}{3a}}+\overset{(D)}{\frac{x^3}{4a^2}}+\overset{(E)}{\frac{x^4}{5a^3}},\ \&c.\ \text{into}\ \overset{(P)}{a}-\overset{(Q)}{\tfrac{1}{3}x}+\overset{(R)}{\frac{x^2}{5a}}-\overset{(S)}{\frac{x^3}{7a^2}}+\overset{(T)}{\frac{x^4}{9a^3}},\ \&c.$$

$$\begin{aligned}=a^2+x\tfrac{1}{2}a+\tfrac{1}{3}x^2+\frac{x^3}{4a}&+\frac{x^4}{5a^2},\ \&c.=a^2+\tfrac{1}{6}ax+\tfrac{11}{30}x^2+\frac{121x^3}{1260a^2}+\frac{281x^4}{1260a^2},\ \&c.\\ -\tfrac{1}{3}ax-\tfrac{1}{6}x^2-\frac{x^3}{9a}&-\frac{x^4}{12a^2}\\ +\tfrac{1}{5}x^2+\frac{x^3}{10a}&+\frac{x^4}{15a^2}\\ -\frac{x^3}{7a}&-\frac{x^4}{14a^2}\\ &+\frac{x^4}{9a^2}\end{aligned}$$

And ſo in all other caſes.

II. From the ſame Equations above we ſhall have $A=\frac{\alpha}{P}$, $B=\frac{\beta-AQ}{P}$, $C=\frac{\gamma-BQ-AR}{P}$, $D=\frac{\delta-CQ-BR-AS}{P}$, $E=\frac{\varepsilon-DQ-CR-BS-AT}{P}$, &c. And then by Subſtitution $\frac{\alpha+\beta+\gamma+\delta+\varepsilon,\ \&c.}{P+Q+R+S+T,\ \&c.}=(A+B+C+D+E,\ \&c.=)\ \frac{\alpha}{P}+\frac{\beta-AQ}{P}+\frac{\gamma-BQ-AR}{P}+\frac{\delta-CQ-BR-AS}{P}+\frac{\varepsilon-DQ-CR-BS-AT}{P}$, &c. This Theorem will ſerve commodiouſly for the Diviſion of one infinite Series by another. Here for conveniency-ſake the Capitals A, B, C, D, &c. are retained in the Theorem, to denote the firſt, ſecond, third, fourth, &c. Terms of the Series reſpectively.

Thus, for Example, if we would divide the Series $\overset{(\alpha)}{a^2}+\overset{(\beta)}{\tfrac{1}{6}ax}+\overset{(\gamma)}{\tfrac{11}{30}x^2}+\overset{(\delta)}{\frac{121x^3}{1260a}}+\overset{(\varepsilon)}{\frac{281x^4}{1260a^2}}$, &c. by the Series $\overset{(P)}{a}+\overset{(Q)}{\tfrac{1}{2}x}+\overset{(R)}{\frac{x^2}{3a}}+\overset{(S)}{\frac{x^3}{4a^2}}+\overset{(T)}{\frac{x^4}{5a^3}}$, &c. the Quotient will be $a+\frac{\frac{1}{6}ax-\frac{1}{2}aA}{a}+\frac{\frac{11}{30}x^2-\frac{x^2}{3a}A-\frac{1}{2}xB}{a}+\frac{\frac{121x^3}{1260a^2}-\frac{x^3}{4a^2}A-\frac{x^2}{3a}B-\frac{1}{2}xC}{a}$, &c. Or reſtoring the Values of A, B, C, D, &c. which repreſent the ſeveral Terms as they ſtand in order, the Quotient will become $a-\tfrac{1}{3}x+\frac{x^2}{5a}-\frac{x^3}{7a^2}+\frac{x^4}{9a^3}$, &c. And after the ſame manner in all other Examples.

III. In the laſt Theorem make $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 0$, &c. then $\frac{1}{P+Q+R+S+T,\&c.} = \frac{1}{P} - \frac{AQ}{P} - \frac{BQ+AR}{P} - \frac{CQ+BR+AS}{P} - \frac{DQ+CR+BS+AT}{P}$, &c. which Theorem will readily find the Reciprocal of any infinite Series. Here A, B, C, D, &c. denote the ſeveral Terms of the Series in order, as before.

Thus if we would know the Reciprocal of the Series $\overset{(P)}{a} + \overset{(Q)}{\tfrac{1}{2}x} + \overset{(R)}{\frac{x^2}{3a}} + \overset{(S)}{\frac{x^3}{4a^2}} + \overset{(T)}{\frac{x^4}{5a^3}}$, &c. we ſhall have by Subſtitution $\frac{1}{a} - \frac{\frac{1}{2}xA}{a} - \frac{\frac{x^2}{3a}A + \frac{1}{2}xB}{a} - \frac{\frac{x^3}{4a^2}A + \frac{x^2}{3a}B + \frac{1}{2}xC}{a} - \frac{\frac{x^4}{5a^3}A + \frac{x^3}{4a^2}B + \frac{x^2}{3a}C + \frac{1}{2}xD}{a}$, &c. And reſtoring the Values of A, B, C, D, &c. it will be $\frac{1}{a} - \frac{x}{2a^2} - \frac{x^2}{12a^3} + \frac{x^3}{8a^4} - \frac{79x^4}{720a^5}$, &c. for the Reciprocal required.

Ex. 2. $\frac{1}{1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{16}x^3, \&c.} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{8}x^3$, &c. And ſo of others.

IV. In the firſt Theorem if we make $P = A$, $Q = B$, $R = C$, $S = D$, &c. that is, if we make both to be the ſame Series; we ſhall have

$$\overline{A+B+C+D+E+F+G,\&c.}|^2 = \begin{array}{llllll} A^2 + 2AB & + 2AC & + 2AD & + 2AE & + 2AF & + 2AG,\&c. \\ & + B^2 & + 2BC & + 2BD & + 2BE & + 2BF \\ & & & + C^2 & + 2CD & + 2CE \\ & & & & & + D^2 \end{array}$$

which will be a Theorem for finding the Square of any infinite Series.

Ex. 1. $$\overline{\frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9},\&c.}|^2 = \begin{array}{l} \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{5x^{10}}{128a^8} + \frac{7x^{12}}{256a^{10}},\&c. \\ \qquad\qquad + \frac{x^8}{64a^6} - \frac{x^{10}}{64a^8} + \frac{5x^{12}}{512a^8} \\ \qquad\qquad\qquad\qquad\quad + \frac{x^{12}}{256a^{10}} \end{array}$$

$$= \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{5x^8}{64a^6} - \frac{7x^{10}}{128a^8} + \frac{21x^{12}}{512a^{10}}, \&c.$$

Ex. 2. $\overline{-\frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}, \&c.}|^2 = \frac{1}{4}x^3 + \frac{1}{8}x^4 + \frac{5}{16}x^5$, &c.

Ex. 3. $$\left.\begin{array}{l} \frac{bx}{2a} - \frac{x^2}{2a} + \frac{bx^3}{4a^3}, \&c. \\ \quad - \frac{b^2x^2}{8a^3} + \frac{b^3x^3}{16a^5} \end{array}\right|^2 = \begin{array}{l} \frac{b^2x^2}{4a^2} - \frac{bx^3}{2a^2} + \frac{x^4}{4a^2}, \&c. \\ \qquad\quad - \frac{b^3x^3}{8a^4} + \frac{3b^2x^4}{8a^4} \\ \qquad\qquad\quad + \frac{17b^4x^4}{64a^4} \end{array}$$

Ex. 4. $$\overline{\frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}},\&c.}\Big|^2 = \frac{x^6}{9a^4} - \frac{2x^9}{27a^7} + \frac{13x^{12}}{243a^{10}} - \frac{30x^{15}}{729a^{13}},\&c.$$

V. In this laſt Theorem, if we make $A^2 = P$, $2AB = Q$, $2AC + B^2 = R$, $2AD + 2BC = S$, $2AE + 2BD + C^2 = T$, &c. we ſhall have $A = P^{\frac{1}{2}}$, $B = \frac{Q}{2A}$, $C = \frac{R - B^2}{2A}$, $D = \frac{S - 2BC}{2A}$, $E = \frac{T - 2BD - C^2}{2A}$, &c. Or $\overline{P + Q + R + S + T + U, \&c.}|^{\frac{1}{2}} = P^{\frac{1}{2}} + \frac{Q}{2A} + \frac{R - B^2}{2A} + \frac{S - 2BC}{2A} + \frac{T - 2BD - C^2}{2A} + \frac{U - 2BE - 2CD}{2A}$, &c. By this Theorem the Square-root of any infinite Series may eaſily be extracted. Here A, B, C, D, &c. will repreſent the ſeveral Terms of the Series as they are in ſucceſſion.

Ex. 1. $\overline{x^2 - 2ax + 2a^2 - \frac{a^3}{x} * + \frac{a^5}{4x^3}, \&c.}|^{\frac{1}{2}} = x - a + \frac{a^2}{2x} * - \frac{a^4}{8x^3} *$ &c.

Ex. 2. $\overline{\frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{5x^8}{64a^6} - \frac{7x^{10}}{128a^8} + \frac{21x^{12}}{512a^{10}}, \&c.}|^{\frac{1}{2}} = \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}$, &c.

VI. Becauſe it is by the fourth Theorem

$$\overline{\alpha + \beta + \gamma + \delta + \varepsilon, \&c.}|^2 = \begin{array}{l} \alpha^2 + 2\alpha\beta + 2\alpha\gamma + 2\alpha\delta + 2\alpha\varepsilon, \&c. \\ \qquad\qquad\;\; + \beta^2 \;\; + 2\beta\gamma + 2\beta\delta \\ \qquad\qquad\qquad\qquad\qquad + \gamma^2 \end{array}$$

in the third Theorem for P, Q, R, S, T, &c. write α^2, $2\alpha\beta$, $2\alpha\gamma + \beta^2$, $2\alpha\delta + 2\beta\gamma$, $2\alpha\varepsilon + 2\beta\delta + \gamma^2$, &c. reſpectively. Then

$$\frac{1}{\overline{\alpha + \beta + \gamma + \delta + \varepsilon \; \&c.}|^2} = \frac{1}{\alpha^2} - \frac{2\alpha\beta A}{\alpha^2} - \frac{2\alpha\beta B + \overline{2\alpha\gamma + \beta^2} \times A}{\alpha^2} - \frac{2\alpha\beta C + \overline{2\alpha\gamma + \beta^2} \times B + \overline{2\alpha\delta + 2\beta\gamma} \times A}{\alpha^2}, \&c.$$

And this will be a Theorem for finding the Reciprocal of the Square of any infinite Series. Here A, B, C, D, &c. ſtill denote the Terms of the Series in their order.

VII. If in the firſt Theorem for P, Q, R, S, &c. we write A^2, $2AB$, $2AC + B^2$, $2AD + 2BC$, &c. reſpectively, (that is $\overline{A + B + C + D, \&c.}|^2$, by Theor. 4.) we ſhall have

$$\overline{A + B + C + D + E + F, \&c.}|^3 = \begin{array}{llllll} A^3 + 3A^2B & + 3AB^2 & + 3A^2D & + 3AC^2 & + 3BC^2, & \&c. \\ & + 3A^2C & + 6ABC & + 3B^2C & + 3B^2D & \\ & & + B^3 & + 6ABD & + 6ACD & \\ & & & + 3A^2E & + 6ABE & \\ & & & & + 3AF & \end{array}$$

which will readily give the Cube of any infinite Series.

$$\text{Ex. 1. } \overline{\frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}}, \&c.}|^3 = \begin{array}{l} \frac{x^9}{27a^6} - \frac{x^{12}}{27a^9} + \frac{x^{15}}{81a^{12}} - \frac{10x^{18}}{729a^{15}}, \&c. \\ \qquad\qquad\qquad\quad + \frac{5x^{15}}{243a^{12}} - \frac{10x^{18}}{729a^{15}} \\ \qquad\qquad\qquad\qquad\qquad\quad - \frac{x^{18}}{729a^{15}} \end{array}$$

$$= \frac{x^9}{27a^6} - \frac{x^{12}}{27a^9} + \frac{8x^{15}}{243a^{12}} - \frac{7x^{18}}{243a^{15}}, \&c.$$

Ex. 2. $\overline{\tfrac{1}{2}x^2+\tfrac{1}{12}x^3+\tfrac{1}{216}x^4,\&c.}\,|^3=\tfrac{1}{8}x^6+\tfrac{1}{24}x^7+\tfrac{1}{288}x^8$, &c.

VIII. In the laſt Theorem, if we make $A^3=P$, $3A^2B=Q$, $AB^2+3A^2C=R$, $B^3+6ABC+3A^2D=S$, &c. then $A=P^{\frac{1}{3}}$, $B=\dfrac{Q}{3A^2}$, $C=\dfrac{R-3AB^2}{3A^2}$, $D=\dfrac{S-6ABC-B^3}{3A^2}$, &c. that is $\overline{P+Q+R+S+T,\&c.}\,|^{\frac{1}{3}}=P^{\frac{1}{3}}+\dfrac{Q}{3A^2}+\dfrac{R-3AB^2}{3A^2}+\dfrac{S-B^3-6ABC}{3A^2}+\dfrac{T-3AC^2-3B^2C-6ABD}{3A^2}$, &c. And by this Theorem the Cube-root of any infinite Series may be extracted. Here alſo A, B, C, D, &c. will repreſent the Terms as they ſtand in order.

Ex. 1. $\overline{\dfrac{x^9}{27a^6}-\dfrac{x^{12}}{27a^9}+\dfrac{8x^{15}}{243a^{12}}-\dfrac{7x^{18}}{243a^{15}},\&c.}\,\Big|^{\frac{1}{3}}=\dfrac{x^3}{3a^2}-\dfrac{x^6}{9a^5}+\dfrac{5x^9}{81a^8}-\dfrac{10x^{12}}{243a^{11}},\&c.$

Ex. 2. $\overline{\tfrac{1}{8}x^6+\tfrac{1}{24}x^7+\tfrac{1}{288}x^8,\&c.}\,|^{\frac{1}{3}}=\tfrac{1}{2}x^2+\tfrac{1}{12}x^3+\tfrac{1}{216}x^4$, &c.

IX. Becauſe it is by the ſeventh Theorem $\overline{\alpha+\beta+\gamma+\delta,\&c.}\,|^3$

$$=\begin{array}{l}\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3,\ \&c.\\ \qquad\qquad\ \ +3\alpha^2\gamma+6\alpha\beta\gamma\\ \qquad\qquad\qquad\qquad\ +3\alpha^2\delta\end{array}$$

in the third Theorem for P, Q, R, S, T, &c. write α^3, $3\alpha^2\beta$, $3\alpha\beta^2+3\alpha^2\gamma$, $\beta^3+6\alpha\beta\gamma+3\alpha^2\delta$, $3\alpha\gamma^2+3\beta^2\gamma+6\alpha\beta\delta+3\alpha^2\varepsilon$, &c. reſpectively; then $\dfrac{1}{\overline{\alpha+\beta+\gamma+\delta,\&c.}\,|^3}$

$$=\frac{1}{\alpha^3}-\frac{3\alpha^2\beta A}{\alpha^3}-\frac{3\alpha^2\beta B+\overline{3\alpha\beta^2+3\alpha^2\gamma}\times A}{\alpha^3}-\frac{3\alpha^2\beta C+\overline{3\alpha\beta^2+3\alpha^2\gamma}\times B+\overline{\beta^3+6\alpha\beta\gamma+3\alpha\delta}\times A\ \&c.}{\alpha^3}$$

This Theorem will give the Reciprocal of the Cube of any infinite Series; where A, B, C, D, &c. ſtand for the Terms in order.

X. Laſtly, in the firſt Theorem if we make $P=A^4$, $Q=3A^2B$, $R=3AB^2+3A^2C$, $S=B^3+6ABC+3A^2D$, &c. we ſhall have

$$\overline{A+B+C+D,\&c.}\,|^4=\begin{array}{l}A^4+4A^3B+6A^2B^2+4AB^3,\ \&c.\\ \qquad\qquad\quad\ +4A^3C+12A^2BC\\ \qquad\qquad\qquad\qquad\quad\ +4A^3D\end{array}$$

which will be a Theorem for finding the Biquadrate of any infinite Series.

And thus we might proceed to find particular Theorems for any other Powers or Roots of any infinite Series, or for their Reciprocals, or any fractional Powers compounded of theſe; all which will be found very convenient to have at hand, continued to a competent number of Terms, in order to facilitate the following Operations. Or it may be ſufficient to lay before you the elegant and general Theorem, contrived for this purpoſe, by that ſkilful Mathematician, and my good Friend, the ingenious Mr. *A. De Moivre*, which was firſt publiſh'd in the Philoſophical Tranſactions, N° 230, and which will readily perform all theſe Operations.

Or we may have recourſe to a kind of Mechanical Artifice, by which all the foregoing Operations may be perform'd in a very eaſy and general manner, as here follows.

When two infinite Series are to be multiply'd together, in order to find a third which is to be their Product, call one of them the Multiplicand, and the other the Multiplier. Write down upon your Paper the Terms of the Multiplicand, with their Signs, in a deſcending order, ſo that the Terms may be at equal diſtances, and juſt under one another. This you may call your fixt or right-hand Paper. Prepare another Paper, at the right-hand Edge of which write down the Terms of the Multiplier, with their proper Signs, in an aſcending Order, ſo that the Terms may be at the ſame equal diſtances from each other as in the Multiplicand, and juſt over one another. This you may call your moveable or left-hand Paper. Apply your moveable Paper to your fixt Paper, ſo that the firſt Term of your Multiplier may ſtand over-againſt the firſt Term of your Multiplicand. Multiply theſe together, and write down the Product in its place, for the firſt Term of the Product required. Move your moveable Paper a ſtep lower, ſo that two of the firſt Terms of the Multiplier may ſtand over-againſt two of the firſt Terms of the Multiplicand. Find the two Products, by multiplying each pair of the Terms together, that ſtand over-againſt one another; abbreviate them if it may be done, and ſet down the Reſult for the ſecond Term of the Product required. Move your moveable Paper a ſtep lower, ſo that three of the firſt Terms of the Multiplier may ſtand over-againſt three of the firſt Terms of the Multiplicand. Find the three Products, by multiplying each pair of the Terms together that ſtand over-againſt one another; abbreviate them, and ſet down the Reſult for the third Term of the Product. And proceed in the ſame manner to find the fourth, and all the following Terms.

I ſhall illuſtrate this Method by an Example of two Series, taken from the common Scale of Denary or Decimal Arithmetick; which will equally explain the Proceſs in all other infinite Series whatever.

Let the Numbers to be multiply'd be 37,528936, &c. and 528,73041, &c. which, by ſupplying X or 10 where it is underſtood, will become the Series $3X + 7X^0 + 5X^{-1} + 2X^{-2} + 8X^{-3} + 9X^{-4} + 3X^{-5} + 6X^{-6}$, &c. and $5X^2 + 2X + 8X^0 + 7X^{-1} + 3X^{-2} + 0X^{-3} + 4X^{-4} + 1X^{-5}$, &c. and call the firſt the Multiplicand, and the ſecond the Multiplier. Theſe being diſpoſed as is preſcribed, will ſtand as follows.

Multiplier,

Multiplier,	Multiplicand	Product	$1X^4$
&c.	$3X$	$15X^3$	$9X^3$
$+1X^{-5}$	$+7X^0$	$+41X^2$	$8X^2$
$+4X^{-4}$	$+5X^{-1}$	$+63X$	$4X$
$+0X^{-3}$	$+2X^{-2}$	$+97X^0$	$2X^0$
$+3X^{-2}$	$+8X^{-3}$	$+142X^{-1}$	$6X^{-1}$
$+7X^{-1}$	$+9X^{-4}$	$+133X^{-2}$	$8X^{-2}$
$+8X^0$	$+3X^{-5}$	$+138X^{-3}$	$8X^{-3}$
$+2X$	$+6X^{-6}$	$+201X^{-4}$	$1X^{-4}$
$5X^2$	&c.	&c.	&c.

Now the firſt Term of the moveable Paper, or Multiplier, being apply'd to the firſt Term of the Multiplicand, will give $5X^2 \times 3X = 15X^3$ for the firſt Term of the Product. Then the two firſt Terms of each being apply'd together, they will give $5X^2 \times 7X^0 + 2X \times 3X = 41X^2$ for the ſecond Term of the Product. Then the three firſt Terms of each being apply'd together, they will give $5X^2 \times 5X^{-1} + 2X \times 7X^0 + 8X^0 \times 3X = 63X$ for the third Term of the Product. And ſo on. So that the Product required will be $15X^3 + 41X^2 + 63X + 97X^0 + 142X^{-1} + 133X^{-2} + 138X^{-3} + 201X^{-4}$, &c. Now this will be a Number in the Decimal Scale of Arithmetick, becauſe $X = 10$. But in that Scale, when it is regular, the Coefficients muſt always be affirmative Integers, leſs than the Root 10; and therefore to reduce theſe to ſuch, ſet them orderly under one another, as is done here, and beginning at the loweſt, collect them as they ſtand, by adding up each Column. The reaſon of which is this. Becauſe $201X^{-4} = 20X^{-3} + 1X^{-4}$, we muſt ſet down $1X^{-4}$, and add $20X^{-3}$ to the line above. Then becauſe $20X^{-3} + 138X^{-3} = 158X^{-3} = 15X^{-2} + 8X^{-3}$, we muſt ſet down $8X^{-3}$; and add $15X^{-2}$ to the line above. Then becauſe $15X^{-2} + 133X^{-2} = 148X^{-2} = 14X^{-1} + 8X^{-2}$, we muſt ſet down $8X^{-2}$, and add $14X^{-1}$ to the line above. And ſo we muſt proceed through the whole Number. So that at laſt we ſhall find the Product to be $1X^4 + 9X^3 + 8X^2 + 4X + 2X^0 + 6X^{-1} + 8X^{-2} + 8X^{-3}$, &c. Or by ſuppreſſing X, or 10, and leaving it to be ſupply'd by the Imagination, the Product required will be 19842,688, &c.

When one infinite Series is to be divided by another, write down the Terms of the Dividend, with their proper Signs, in a deſcending order, ſo that the Terms may be at equal diſtances, and juſt under one another. This is your fixt or right-hand Paper. Prepare another Paper, at the right-hand Edge of which write down the Terms of the Diviſor in an aſcending order, with all their Signs changed except the firſt, ſo that the Terms may be at the ſame equal diſtances as before, and juſt over one another. This will be your moveable or left-hand Paper. Apply your moveable Paper to your fixt Paper, ſo that the firſt Term of the Diviſor may be over-againſt the firſt Term of the Dividend. Divide the firſt Term of the Dividend by the firſt Term of the Diviſor, and ſet down the Quotient over-againſt them to the right-hand, for the firſt Term of the Quotient required. Move your moveable Paper a ſtep lower, ſo that two of the firſt Terms of the Diviſor may be over-againſt two of the firſt Terms of the Dividend. Collect the ſecond Term of the Dividend, together with the Product of the firſt Term of the Quotient now found, multiply'd by the Terms over-againſt it in the left-hand Paper; theſe divided by the firſt Term of the Diviſor will be the ſecond Term of the Quotient required. Move your moveable Paper a ſtep lower, ſo that three of the firſt Terms of the Diviſor may ſtand over-againſt three of the firſt Terms of the Dividend. Collect the third Term of the Dividend, together with the two Products of the two firſt Terms of the Quotient now found, each being multiply'd into the Term over-againſt it, in the left-hand Paper. Theſe divided by the firſt Term of the Diviſor will be the third Term of the Quotient required. Move your moveable Paper a ſtep lower, ſo that four of the firſt Terms of the Diviſor may ſtand over-againſt four of the firſt Terms of the Dividend. Collect the fourth Term of the Dividend, together with the three Products of the three firſt Terms of the Quotient now found, each being multiply'd by the Term over-againſt it in the left-hand Paper. Theſe divided by the firſt Term of the Diviſor will be the fourth Term of the Quotient required. And ſo on to find the fifth, and the ſucceeding Terms.

For an Example let it be propoſed to divide the infinite Series $a^2 + \frac{1}{6}ax + \frac{11}{80}x^2 + \frac{121x^3}{1260a} + \frac{281x^4}{1260a^2}$, &c. by the Series $a + \frac{1}{2}x + \frac{x^2}{3a} + \frac{x^3}{4a^2} + \frac{x^4}{5a^3}$, &c. Theſe being diſpoſed as is preſcribed, will ſtand as here follows.

Diviſor,

Diviſor, &c.	Dividend		Quotient
	a^2	a -	a
$-\frac{x^4}{5a^3}$	$+\frac{1}{6}ax$	$+\frac{1}{6}ax - \frac{1}{2}ax = -\frac{1}{3}ax$ - - - - - - - -	$-\frac{1}{3}x$
$-\frac{x^3}{4a^2}$	$+\frac{11}{30}x^2$	$+\frac{11}{30}x^2 + \frac{1}{6}x^2 - \frac{1}{3}x^2 = +\frac{1}{5}x^2$ - - - -	$+\frac{x^2}{5a}$
$-\frac{x^2}{3a}$	$+\frac{121x^3}{1260a}$	$+\frac{121x^3}{1260a} \frac{x^3}{10a} + \frac{x^3}{9a} - \frac{x^3}{4a} = -\frac{x^3}{7a}$ - - -	$-\frac{x^3}{7a^2}$
$-\frac{1}{2}x$	$+\frac{281x^4}{1260a^2}$	$+\frac{281x^4}{1260a^2} + \frac{x^4}{14a^2} - \frac{x^4}{15a^2} + \frac{x^4}{12a^2} - \frac{x^4}{5a^2} = +\frac{x^4}{9a^2}$	$+\frac{x^4}{9a^3}$
a	&c.	&c.	&c.

Here if we apply the firſt Term of the Diviſor a, to the firſt Term of the Dividend a^2, by Diviſion we ſhall have a for the firſt Term of the Quotient. Then applying the two firſt Terms of the Diviſor to the two firſt Terms of the Dividend, we ſhall have $\frac{1}{6}ax$ to be collected with the Product $a \times -\frac{1}{2}x$, or $-\frac{1}{2}ax$, which will make $-\frac{1}{3}ax$; and this divided by a, the firſt Term of the Diviſor, will give $-\frac{1}{3}x$ for the ſecond Term of the Quotient. And ſo of the other Terms; and in like manner for all other Examples.

When an infinite Series is to be raiſed to any Power, or when any Root of it is to be extracted, it may be perform'd in all caſes by a like Artifice. Prepare your fixt or right-hand Paper, by writing down the natural Numbers 0, 1, 2, 3, 4, &c. juſt under one another at equal diſtances, reſerving places to the right-hand for the ſeveral Terms of the Power or Root, as they ſhall be found. The firſt Term of which Series may be immediately known from the firſt Term of the given Series, and from the given Index of the Power or Root, whether that Index be an Integer or a Fraction, affirmative or negative; and that Term therefore may be ſet down in its place, over-againſt the firſt Number 0. Prepare your moveable or left-hand Paper, by writing down, towards the edge of the Paper at the right-hand, all the Terms of the given Series, except the firſt, over one another in order, at the ſame diſtances as the Numbers in the other Paper. After which, nearer the edge of the Paper, write juſt over one another, firſt the Index of the Power or Root to be found, then its double, then its triple, and ſo the reſt of its multiples, with the negative Sign after each, as far as the Terms of the Series extend. And alſo the firſt Term of the given Series may be wrote below. Thus will the moveable Paper be prepared. Theſe multiples, together with the following negative Signs, and the Numbers

0,

0, 1, 2, 3, 4, &c. on the other Paper, when they meet together, will compleat the numeral Coefficients. Apply therefore the ſecond Term of the moveable Paper to the uppermoſt Term of the fixt Paper, and the Product made by the continual Mutiplication of the three Factors that ſtand in a line over-againſt one another, [which are the ſecond Term of the given Series, the numeral Coefficient, (here the given Index,) and the firſt Term of the Series already found,] divided by the firſt Term of the given Series, will be the ſecond Term of the Series required, which is to be ſet down in its place over-againſt 1. Move the moveable Paper a ſtep lower, and the two Products made by the multiplication of the Factors that ſtand over-againſt one another, (in which, and elſewhere, care muſt be had to take the numeral Coefficients compleat,) divided by twice the firſt Term of the given Series, will be the third Term of the Series required, which is to be ſet down in its place over-againſt 2. Move the moveable Paper a ſtep lower, and the three Products made by the multiplication of the Factors that ſtand over-againſt one another, divided by thrice the firſt Term of the given Series, will be the fourth Term of the Series required. And ſo you may proceed to find the next, and the ſubſequent Terms.

It may not be amiſs to give one general Example of this Reduction, which will comprehend all particular Caſes. If the Series $az + bz^2 + cz^3 + dz^4$, &c. be given, of which we are to find any Power, or to extract any Root; let the Index of this Power or Root be m. Then prepare the moveable or left-hand Paper as you ſee below, where the Terms of the given Series are ſet over one another in order, at the edge of the Paper, and at equal diſtances. Alſo after every Term is put a full point, as a Mark of Multiplication, and after every one, (except the firſt or loweſt) are put the ſeveral Multiples of the Index, as m, $2m$, $3m$, $4m$, &c. with the negative Sign — after them. Likewiſe a *vinculum* may be underſtood to be placed over them, to connect them with the other parts of the numeral Coefficients, which are on the other Paper, and which make them compleat. Alſo the firſt Term of the given Series is ſeparated from the reſt by a line, to denote its being a Diviſor, or the Denominator of a Fraction. And thus is the moveable Paper prepared.

To prepare the fixt or right-hand Paper, write down the natural Numbers 0, 1, 2, 3, 4, &c. under one another, at the ſame equal diſtances as the Terms in the other Paper, with a Point after them as a Mark of Multiplication; and over-againſt the firſt Term o write a^mz^m for the firſt Term of the Series required. The reſt of the Terms are to be wrote down orderly under this, as they ſhall be found, which will be in this manner. To the firſt Term o in the fixt Paper apply the ſecond Term of the moveable Paper, and they will then exhibit this Fraction $\frac{bz^2.\overline{m-0}.\,a^mz^m}{az.\,1}$, which being reduced to this $ma^{m-1}bz^{m+1}$, muſt be ſet down in its place, for the ſecond Term of the Series required. Move the moveable Paper a ſtep lower, and you will have this Fraction exhibited

$$\frac{\begin{array}{l}+cz^3.\overline{2m-0}.\,a^mz^m\\+bz^2.\overline{m-1}.\,ma^{m-1}bz^{m+1}\end{array}}{az.\,2}$$

which being reduced will become $ma^{m-1}c+m\times\frac{m-1}{2}a^{m-2}b^2\times z^{m+2}$, to be put down for the third Term of the Series required. Bring down the moveable Paper a ſtep lower, and you will have the Fraction

$$\frac{\begin{array}{l}+dz^4.\overline{3m-0}.\,a^mz^m\\+cz^3.\overline{2m-1}.\,\underline{ma^{m-1}bz^{m+1}}\\+bz^2.\overline{m-2}.\,ma^{m-1}c+m\times\frac{m-1}{2}a^{m-2}b^2\times z^{m+2}\end{array}}{az.\,3}$$

which reduc'd will be $ma^{m-1}d+m\times\frac{m-1}{1}a^{m-2}bc+m\times\frac{m-1}{2}\times\frac{m-2}{3}a^{m-3}b^3\times z^{m+3}$, for the fourth Term of the Series required. And in the ſame manner are all the reſt of the Terms to be found.

Moveable Paper, &c.	Fixt Paper
$+dz^4.\overline{3m-}$	0. a^mz^m
$+cz^3.\overline{2m-}$	1. $ma^{m-1}bz^{m+1}$
$+bz^2.\ \overline{m-}$	2. $m\times\frac{m-1}{2}a^{m-2}b^2+ma^{m-1}c\times z^{m+2}$
$az.$	3. $m\times\frac{m-1}{2}\times\frac{m-2}{3}a^{m-3}b^3+m\times\frac{m-1}{1}a^{m-2}bc+ma^{m-1}d\times z^{m+3}$
	&c. &c.

N. B. This Operation will produce Mr. *De Moivre*'s Theorem mentioned before, the Inveſtigation of which may be ſeen in the place there quoted, and ſhall be exhibited here in due time and place. And this therefore will ſufficiently prove the truth of the preſent Proceſs. In particular Examples this Method will be found very eaſy and practicable.

But now to ſhew ſomething of the uſe of theſe Theorems, and at the ſame time to prepare the way for the Solution of Affected and Fluxional Equations; we will here make a kind of retroſpect, and reſume our Author's Examples of ſimple Extractions, beginning with Diviſion itſelf, which we ſhall perform after a different and an eaſier manner.

Thus to divide aa by $b+x$, or to reſolve the Fraction $\frac{aa}{b+x}$ into a Series of ſimple Terms; make $\frac{aa}{b+x}=y$, or $by+xy=aa$. Now to find the quantity y diſpoſe the Terms of this Equation after this manner $\left.\begin{array}{l} by \\ +xy \end{array}\right\} = a^2$, and proceed in the Reſolution as you ſee is done here.

$$\begin{array}{r} \left.\begin{array}{r} by \\ +xy \end{array}\right\} \begin{array}{l} = a^2 - \frac{a^2x}{b} + \frac{a^2x^2}{b^2} - \frac{a^2x^3}{b^3} + \frac{a^2x^4}{b^4}, \text{ \&c.} \\ - - - + \frac{a^2x}{b} - \frac{a^2x^2}{b^2} + \frac{a^2x^3}{b^3} - \frac{a^2x^4}{b^4}, \text{ \&c.} \end{array} \\ y = \frac{a^2}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} + \frac{a^2x^4}{b^5}, \text{ \&c.} \end{array}$$

Here by the diſpoſition of the Terms a^2 is made the firſt Term of the Series belonging (or equivalent) to by, and therefore dividing by b, $\frac{a^2}{b}$ will be the firſt Term of the Series equivalent to y, as is ſet down below. Then will $+\frac{a^2x}{b}$ be the firſt Term of the Series $+xy$, which is therefore ſet down over-againſt it; as alſo it is ſet down over-againſt by, but with a contrary Sign, to be the ſecond Term of that Series. Then will $-\frac{a^2x}{b^2}$ be the ſecond Term of y, to be ſet down in its place, which will give $-\frac{a^2x^2}{b^2}$ for the ſecond Term of $+xy$; and this with a contrary Sign muſt be ſet down for the third Term of by. Then will $+\frac{a^2x^2}{b^3}$ be the third Term of y, and therefore $+\frac{a^2x^3}{b^3}$ will be the third Term of $+xy$, which with a contrary Sign muſt be made the fourth Term of by, and therefore $-\frac{a^2x^3}{b^4}$ will be the fourth Term of y. And ſo on for ever.

Now the *Rationale* of this Proceſs, and of all that will here follow of the ſame kind, may be manifeſt from theſe Conſiderations. The unknown Terms of the Equation, or thoſe wherein y is found, are (by the *Hypotheſis*) equal to the known Term aa. And each of thoſe unknown Terms is reſolved into its equivalent Series, the Aggregate of which muſt ſtill be equal to the ſame known Term aa; (or perhaps Terms.) Therefore all the ſubſidiary and adventitious Terms, which are introduced into the Equation to aſſiſt the Solution, (or the Supplemental Terms,) muſt mutually deſtroy one another.

Or we may reſolve the ſame Equation in the following manner:

$$\left.\begin{array}{r} by \\ +xy \end{array}\right\} \begin{array}{l} = - - - + \frac{ba^2}{x} - \frac{b^2a^2}{x^2} + \frac{b^3a^2}{x^3}, \&c. \\ = a^2 - \frac{ba^2}{x} + \frac{b^2a^2}{x^2} - \frac{b^3a^2}{x^3}, \&c. \end{array}$$

$$y = \frac{a^2}{x} - \frac{ba^2}{x^2} + \frac{b^2a^2}{x^3} - \frac{b^3a^2}{x^4}, \&c.$$

Here a^2 is made the firſt Term of $+xy$, and therefore $\frac{a^2}{x}$ muſt be put down for the firſt Term of y. This will give $+\frac{ba^2}{x}$ for the firſt Term of by, which with a contrary Sign muſt be the ſecond Term of $+xy$, and therefore $-\frac{ba^2}{x^2}$ muſt be put down for the ſecond Term of y. Then will $-\frac{b^2a^2}{x^2}$ be the ſecond Term of by, which with a contrary Sign will be the third Term of $+xy$, and therefore $+\frac{b^2a^2}{x^3}$ will be the third Term of y. And ſo on. Therefore the Fraction propoſed is reſolved into the ſame two Series as were found above.

If the Fraction $\frac{1}{1+x^2}$ were given to be reſolved, make $\frac{1}{1+x^2} = y$, or $y + x^2y = 1$, the Reſolution of which Equation is little more than writing down the Terms, in the manner following:

$$\left.\begin{array}{r} y \\ +x^2y \end{array}\right\} \begin{array}{l} = 1 - x^2 + x^4 - x^6 + x^8, \&c. \\ - - - + x^2 - x^4 + x^6, \&c. \end{array} \qquad \left.\begin{array}{r} y \\ +x^2y \end{array}\right\} \begin{array}{l} - - - + x^{-2} - x^{-4} + x^{-6} - x^{-8}, \&c. \\ = 1 - x^{-2} + x^{-4} - x^{-6}, \&c. \end{array}$$

Here in the firſt Paradigm, as 1 is made the firſt Term of y, ſo will x^2 be the firſt Term of x^2y, and therefore $-x^2$ will be the ſecond Term of y, and therefore $-x^4$ will be the ſecond Term of x^2y, and therefore $+x^4$ will be third Term of y; &c. Alſo in the ſecond Paradigm, as 1 is made the firſt Term of x^2y, ſo will $+x^{-2}$ be the firſt Term of y, and therefore $-x^{-2}$ will be the ſecond Term of x^2y, or $-x^{-4}$ will be the ſecond Term of y; &c.

To reſolve the compound Fraction $\frac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}$ into ſimple Terms, make $\frac{2x^{\frac{1}{2}}-x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}=y$, or $2x^{\frac{1}{2}}-x^{\frac{3}{2}}=y+x^{\frac{1}{2}}y-3xy$; which Equation may be thus reſolved:

$$\begin{array}{r|l}
 & = 2x^{\frac{1}{2}} \quad * \quad - x^{\frac{3}{2}} \\
\hline
y & \text{- - -}\, 2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^{2} + 34x^{\frac{5}{2}} - 73x^{3}, \&c. \\
+x^{\frac{1}{2}}y & \text{- - - - -} + 2x - 2x^{\frac{3}{2}} + 7x^{2} - 13x^{\frac{5}{2}} + 34x^{3}, \&c. \\
-3xy & \text{- - - - - - - - -} - 6x^{\frac{3}{2}} + 6x^{2} - 21x^{\frac{5}{2}} + 39x^{3}, \&c.
\end{array}$$

Place the Terms of the Equation, in which the unknown quantity y is found, in a regular deſcending order, and the known Terms above, as you ſee is done here. Then bring down $2x^{\frac{1}{2}}$ to be the firſt Term of y, which will give $+2x$ for the firſt Term of the Series $+x^{\frac{1}{2}}y$, which muſt be wrote with a contrary Sign for the ſecond Term of y. Then will the ſecond Term of $+x^{\frac{1}{2}}y$ be $-2x^{\frac{3}{2}}$, and the firſt Term of the Series $-3xy$ will be $-6x^{\frac{3}{2}}$, which together make $-8x^{\frac{3}{2}}$. And this with a contrary Sign would have been wrote for the third Term of y, had not the Term $-x^{\frac{3}{2}}$ been above, which reduces it to $+7x^{\frac{3}{2}}$ for the third Term of y. Then will $+7x^{2}$ be the third Term of $+x^{\frac{1}{2}}y$, and $+6x^{2}$ will be the ſecond Term of $-3ay$, which being collected with a contrary Sign, will make $-13x^{2}$ for the fourth Term of y; and ſo on, as in the Paradigm.

If we would reſolve this Fraction, or this Equation, ſo as to accommodate it to the other caſe of convergency, we may invert the Terms, and proceed thus:

$$\begin{array}{r|l}
 & = -x^{\frac{3}{2}} \quad * \quad + 2x^{\frac{1}{2}} \\
\hline
-3xy & \text{- - - -} - x^{\frac{3}{2}} - \tfrac{1}{3}x + \tfrac{14}{9}x^{\frac{1}{2}} + \tfrac{11}{27}, \&c. \\
+x^{\frac{1}{2}}y & \text{- - - - - - -} + \tfrac{1}{3}x + \tfrac{1}{9}x^{\frac{1}{2}} - \tfrac{14}{27}, \&c. \\
+y & \text{- - - - - - - - - - -} + \tfrac{1}{3}x^{\frac{1}{2}} + \tfrac{1}{9}, \&c.
\end{array}$$

$$y=\tfrac{1}{3}x^{\frac{1}{2}}+\tfrac{1}{9}-\tfrac{14}{27}x^{-\frac{1}{2}}-\tfrac{11}{81}x^{-1}, \&c.$$

Bring down $-x^{\frac{3}{2}}$ to be the firſt Term of $-3xy$, whence $+\frac{1}{3}x^{\frac{1}{2}}$ will be the firſt Term of y, to be ſet down in its place. Then the firſt

$$\left.\begin{matrix} by \\ +.xy \end{matrix}\right\} = \begin{matrix} a^2. + a + b + c + d + e + \&c \\ + \alpha. + \beta + \gamma + \delta + \varepsilon + \&c \end{matrix}$$

$$y = A + B + C + D + E + \&c$$

$$b^c y = a^2$$

$$y = \frac{a^2}{b}$$

~~[illegible]~~ $\alpha (= xy) = \frac{a^2 x}{b}$

~~[illegible]~~

$$by\,(= a^2 - xy) = a^2 - \frac{a^2 x}{b} = a^2 + a$$

$$a = \frac{-a^2 x}{b}$$

$$\left.\begin{matrix} by \\ +xy \end{matrix}\right\} = \begin{matrix} a^2 - b + d - e + f - \&c \\ \alpha - \beta + \gamma - \delta + \&c \end{matrix}$$

$$\left.\begin{matrix} b.y \\ x.y \end{matrix}\right\} = a^2. - a$$

$$y = \frac{a^2}{b}$$

$$\alpha\,(= xy) = \frac{a^2 x}{b}$$

$$by\,(= a^2 - xy) = a^2 - \alpha = a^2 - c$$

$$c = \alpha$$

$$\left.\begin{matrix} c.y \\ c^2.y \\ c3(y \end{matrix}\right\} = a^2$$

$$by = a^2 - c = a^2 - \alpha$$

$$y = \frac{a^2 - \alpha}{b}$$

$$xy = \left(\frac{a^2 - \alpha}{b}\right)x = \frac{a^2 x}{b} - \frac{\alpha}{b}x = \alpha - \frac{\alpha}{b}x = \alpha - \beta$$

$$\beta = \frac{\alpha x}{b}$$

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firſt Term of $+x^{\frac{1}{2}}y$ will be $+\frac{1}{3}x$, which with a contrary Sign will be the ſecond Term of $-3xy$, and therefore $+\frac{1}{9}$ will be the ſecond Term of y. Then the ſecond Term of $+x^{\frac{1}{2}}y$ will be $+\frac{1}{9}x^{\frac{1}{2}}$, and the firſt Term of y being $+\frac{1}{3}x^{\frac{1}{2}}$, theſe two collected with a contrary Sign would have made $-\frac{4}{9}x^{\frac{1}{2}}$ for the third Term of $-3xy$, had not the Term $+2x^{\frac{1}{2}}$ been preſent above. Therefore uniting theſe, we ſhall have $+\frac{14}{9}x^{\frac{1}{2}}$ for the third Term of $-3xy$, which will give $-\frac{14}{27}x^{-\frac{1}{2}}$ for the third Term of y. Then will the third Term of $+x^{\frac{1}{2}}y$ be $-\frac{14}{27}$, and the ſecond Term of y being $+\frac{1}{9}$, theſe two collected with a contrary Sign will make $+\frac{11}{27}$ for the fourth Term of $-3xy$, and therefore $-\frac{11}{81}x^{-1}$ will be the fourth Term of y; and ſo on.

And thus much for Diviſion; now to go on to the Author's pure or ſimple Extractions.

To find the Square-root of $aa+xx$, or to extract the Root y of this Equation $yy=aa+xx$; make $y=a+p$, then we ſhall have by Subſtitution $2ap+pp=xx$, of which affected Quadratick Equation we may thus extract the Root p. Diſpoſe the Terms in this manner $\left.\begin{matrix}2ap\\+pp\end{matrix}\right\}=xx$, the unknown Terms in a deſcending order on one ſide, and the known Term or Terms on the other ſide of the Equation, and proceed in the Extraction as is here directed.

$$\left.\begin{matrix}2ap\\+p^2\end{matrix}\right\}\begin{array}{l}=x^2-\frac{x^4}{4a^2}+\frac{x^6}{8a^4}-\frac{5x^8}{64a^6}+\frac{7x^{10}}{128a^8},\ \&c.\\ \cdots\ +\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{5x^8}{64a^6}-\frac{7x^{10}}{128a^8},\ \&c.\end{array}$$

$$p=\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\frac{7x^{10}}{256a^9},\ \&c.$$

By this Diſpoſition of the Terms, x^2 is made the firſt Term of the Series belonging to $2ap$; then we ſhall have $\frac{x^2}{2a}$ for the firſt Term of the Series p, as here ſet down underneath. Therefore $\frac{x^4}{4aa}$ will be the firſt Term of the Series p^2, to be put down in its place over-againſt p^2. Then, by what is obſerved before, it muſt be put down with a contrary Sign as the ſecond Term of $2ap$, which will make the ſecond Term of p to be $-\frac{x^4}{8a^3}$. Having there-

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fore the two firſt Terms of $p = \frac{x^2}{2a} - \frac{x^4}{8a^3}$, we ſhall have, (by any of the foregoing Methods for finding the Square of an infinite Series,) the two firſt Terms of $p^2 = \frac{x^4}{4a^2} - \frac{x^6}{8a^4}$; which laſt Term muſt be wrote with a contrary Sign, as the third Term of $2ap$. Therefore the third Term of p is $\frac{x^6}{16a^5}$, and the third Term of p^2 (by the aforeſaid Methods) will be $\frac{5x^8}{64a^6}$, which is to be wrote with a contrary Sign, as the fourth Term of $2ap$. Then the fourth Term of p will be $-\frac{5x^8}{128a^7}$, and therefore the fourth Term of p^2 is $-\frac{7x^{10}}{128a^8}$, which is to be wrote with a contrary Sign for the fifth Term of $2ap$. This will give $\frac{7x^{10}}{256a^9}$ for the fifth Term of p; and ſo we may proceed in the Extraction as far as we pleaſe.

Or we may diſpoſe the Terms of the Supplemental Equation thus:

$$\left.\begin{array}{r} 2ap \\ +p^2 \end{array}\right\} \begin{array}{l} ----+2ax-2a^2+\frac{a^3}{x} * -\frac{a^5}{4x^3}, \&c. \\ = x^2-2ax+2a^2-\frac{a^3}{x} * +\frac{a^5}{4x^3}, \&c. \end{array}$$

$$p = x - a + \frac{a^2}{2x} * - \frac{a^4}{8x^3} *, \&c.$$

Here x^2 is made the firſt Term of the Series p^2, and therefore x, (or elſe $-x$,) will be the firſt Term of p. Then $2ax$ will be the firſt Term of $2ap$, and therefore $-2ax$ will be the ſecond Term of p^2. So that becauſe $p^2 = x^2 - 2ax$, &c. by extracting the Square-root of this Series by any of the foregoiug Methods, it will be found $p = x - a$, &c. or $-a$ will be the ſecond Term of the Root p. Therefore the ſecond Term of $2ap$ will be $-2a^2$, which muſt be wrote with a contrary Sign for the third Term of p^2, and thence (by Extraction) the third Term of p will be $\frac{a^2}{2x}$. This will make the third Term of $2ap$ to be $\frac{a^3}{x}$, which makes the fourth Term of p^2 to be $-\frac{a^3}{x}$, and therefore (by Extraction) 0 will be the fourth Term of p. This makes the fourth Term of $2ap$ to be 0, as alſo of p^2. Then $-\frac{a^4}{8x^3}$ will be the fifth Term of p. Then the fifth Term of

$2ap$ will be $-\frac{a^5}{4x^3}$, which will make the ſixth Term of p^2 to be $\frac{a^5}{4x^3}$; and therefore o will be the ſixth Term of p, &c.

Here the Terms will be alternately deficient; ſo that in the given Equation $yy = aa + xx$, the Root will be $y = a + x - a + \frac{a^2}{2x}$, &c. that is $y = x + \frac{a^2}{2x} - \frac{a^4}{8x^3} + \frac{a^6}{16x^5}$, &c. which is the ſame as if we ſhould change the order of the Terms, or if we ſhould change a into x, and x into a.

If we would extract the Square-root of $aa - xx$; or find the Root y of the Equation $yy = aa - xx$; make $y = a + p$, as before; then $2ap + p^2 = -x^2$, which may be reſolved as in the following Paradigm:

$$\left.\begin{array}{r} 2ap \\ +p^2 \end{array}\right\} \begin{array}{l} = -x^2 - \frac{x^4}{4a^2} - \frac{x^6}{8a^4} - \frac{5x^8}{64a^6} - \frac{7x^{10}}{128a^8}, \text{ \&c.} \\ \text{- - - - - -} + \frac{x^4}{4a^2} + \frac{x^6}{8a^4} + \frac{5x^8}{64a^6} + \frac{7x^{10}}{128a^8}, \text{ \&c.} \end{array}$$

$$p = -\frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \frac{7x^{10}}{256a^9} \text{ \&c.}$$

Here if we ſhould attempt to make $-x^2$ the firſt Term of $+p^2$, we ſhould have $\sqrt{-x^2}$, or $x\sqrt{-1}$, for the firſt Term of p; which being impoſſible, ſhews no Series can be form'd from that Suppoſition.

To find the Square-root of $x - xx$, or the Root y in this Equation $yy = x - xx$, make $y = x^{\frac{1}{2}} + p$, then $x + 2x^{\frac{1}{2}}p + p^2 = x - xx$, or $2x^{\frac{1}{2}}p + p^2 = -x^2$, which may be reſolved after this manner:

$$\left.\begin{array}{r} 2x^{\frac{1}{2}}p \\ +p^2 \end{array}\right\} \begin{array}{l} = -x^2 - \frac{1}{4}x^3 - \frac{1}{8}x^4, \text{ \&c.} \\ \text{- - - - -} + \frac{1}{4}x^3 + \frac{1}{8}x^4, \text{ \&c.} \end{array}$$

$$p = -\frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}, \text{ \&c.}$$

The Terms being rightly diſpoſed, make $-x^2$ the firſt Term of $2x^{\frac{1}{2}}p$; then will $-\frac{1}{2}x^{\frac{3}{2}}$ be the firſt Term of p. Therefore $+\frac{1}{4}x^3$ will be the firſt Term of p^2, which is alſo to be wrote with a contrary Sign for the ſecond Term of $2x^{\frac{1}{2}}p$, which will give $-\frac{1}{8}x^{\frac{5}{2}}$ for the ſecond Term of p. Then (by ſquaring) the ſecond Term of p^2 will be $\frac{1}{8}x^4$, which will give $-\frac{1}{8}x^4$ for the ſecond Term of

$2x^{\frac{1}{2}}p$, and therefore $-\frac{1}{16}x^{\frac{7}{2}}$ for the third Term of p; and ſo on. Therefore in this Equation it will be $y = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$, &c.

So to extract the Root y of this Equation $yy = aa + bx - xx$, make $y = a + p$, then $2ap + p^2 = bx - xx$, which may be thus reſolved.

$$\left.\begin{array}{r} 2ap \\ \\ +p^2 \\ \\ \end{array}\right\} \begin{array}{rrrl} = bx & -x^2 & +\frac{bx^3}{2a^2}, & \&c. \\ & -\frac{b^2x^2}{4a^2} & +\frac{b^3x^3}{8a^4} & \\ - - - - - - - - - & +\frac{b^2x^2}{4a^2} & -\frac{bx^3}{2a^2}, & \&c. \\ & & -\frac{b^3x^3}{8a^4} & \end{array}$$

$$\begin{array}{rrrl} p = \frac{bx}{2a} & -\frac{x^2}{2a} & +\frac{bx^3}{4a^3}, & \&c. \\ & -\frac{b^2x^2}{8a^3} & +\frac{b^3x^3}{16a^5} & \end{array}$$

Make bx the firſt Term of $2ap$; then will $\frac{bx}{2a}$ be the firſt Term of p. Therefore the firſt Term of p^2 will be $+\frac{b^2x^2}{4a^2}$, which is alſo to be wrote with a contrary Sign, ſo that the ſecond Term of $2ap$ will be $-x^2 - \frac{b^2x^2}{4a^2}$, which will make the ſecond Term of p to be $-\frac{x^2}{2a} - \frac{b^2x^2}{8a^3}$. Then by ſquaring, the ſecond Term of p^2 will be $-\frac{bx^3}{2a^2} - \frac{b^3x^3}{8a^4}$, which muſt be wrote with a contrary Sign for the third Term of $2ap$. This will give the third Term of p as in the Example; and ſo on. Therefore the Square-root of the Quantity $a^2 + bx - xx$ will be $a + \frac{bx}{2a} - \frac{x^2}{2a} - \frac{b^2x^2}{8a^3} + \frac{bx^3}{4a^3} + \frac{b^3x^3}{16a^5}$, &c.

Alſo if we would extract the Square-root of $\frac{1 + ax^2}{1 - bx^2}$, we may extract the Roots of the Numerator, and likewiſe of the Denominator, and then divide one Series by the other, as before; but more directly thus. Make $\frac{1 + ax^2}{1 - bx^2} = yy$, or $1 + ax^2 = yy - b^2x^2y^2$. Suppoſe $y = 1 + p$, then $ax^2 = 2p + p^2 - bx^2 - 2bx^2p - bx^2p^2$, which Suppplemental Equation may be thus reſolved.

$$\left.\begin{array}{r|llll}
2p & =ax^2 & +\frac{1}{2}abx^4 & +\frac{3}{8}ab^2x^5, \&c. \\
 & +b & +\frac{3}{4}b^2 & +\frac{5}{8}b^3 \\
 & & -\frac{1}{4}a^2 & -\frac{1}{8}a^2b \\
 & & & +\frac{1}{8}a^3 \\
-2bx^2p & ---- & -ab & -\frac{1}{2}ab^2, \&c. \\
 & & -b^2 & -\frac{3}{4}b^3 \\
 & & & +\frac{1}{4}a^2b \\
+p^2 & ---- & +\frac{1}{4}a^2 & +\frac{1}{8}a^2b, \&c. \\
 & & +\frac{1}{2}ab & +\frac{5}{8}ab^2 \\
 & & +\frac{1}{4}b^2 & -\frac{1}{8}a^3 \\
 & & & +\frac{3}{8}b^3 \\
-bx^2p^2 & -------- & & -\frac{1}{4}a^2b, \&c. \\
 & & & -\frac{1}{2}ab^2 \\
 & & & -\frac{1}{4}b^3
\end{array}\right\}
\qquad
\begin{array}{llll}
p=\frac{1}{2}ax^2 & +\frac{1}{4}abx^4 & +\frac{3}{16}ab^2x^6, \&c. \\
+\frac{1}{2}b & +\frac{3}{8}b^2 & +\frac{5}{16}b^3 \\
 & -\frac{1}{8}a^2 & -\frac{1}{16}a^2b \\
 & & +\frac{1}{16}a^3
\end{array}$$

Make $ax^2 + bx^2$ the firſt Term of $2p$, then will $\frac{1}{2}ax^2 + \frac{1}{2}bx^2$ be the firſt Term of p. Therefore $-abx^4 - b^2x^4$ will be the firſt Term of $-2bx^2p$, and $\frac{1}{4}a^2x^4 + \frac{1}{2}abx^4 + \frac{1}{4}bx^4$ will be the firſt Term of p^2. Theſe being collected, and their Signs changed, muſt be made the ſecond Term of $2p$, which will give $\frac{1}{4}abx^4 + \frac{3}{8}b^2x^4 - \frac{1}{8}a^2x^4$ for the ſecond Term of p. Then the ſecond Term of $-2bx^2p$ will be $-\frac{1}{2}ab^2x^6 - \frac{3}{4}b^3x^6 + \frac{1}{4}a^2bx^6$, and the ſecond Term of p^2 (by ſquaring) will be found $\frac{1}{8}a^2bx^6 + \frac{5}{8}ab^2x^6 - \frac{1}{8}a^3x^6 + \frac{3}{8}b^3x^6$, and the firſt Term of $-bx^2p^2$ will be $-\frac{1}{4}a^2bx^6 - \frac{1}{2}ab^2x^6 - \frac{1}{4}b^3x^6$; which being collected and the Signs changed, will make the third Term of $2p$, half which will be the third Term of p; and ſo on as far as you pleaſe.

And thus if we were to extract the Cube-root of $a^3 + x^3$, or the Root y of this Equation $y^3 = a^3 + x^3$; make $y = a + p$, then by Subſtitution $a^3 + 3a^2p + 3ap^2 + p^3 = a^3 + x^3$, or $3a^2p + 3ap^2 + p^3 = x^3$, which ſupplemental Equation may be thus reſolved.

$$\left.\begin{array}{r|lllll}
3a^2p & = x^3 & -\frac{x^6}{3a^3} & +\frac{5x^9}{27a^6} & -\frac{10x^{12}}{81a^9}, \&c. \\
+3ap^2 & ---- & +\frac{x^6}{3a^3} & -\frac{2x^9}{9a^6} & +\frac{13x^{12}}{81a^9}, \&c. \\
+p^3 & -------- & & +\frac{x^9}{27a^6} & -\frac{x^{12}}{27a^9}, \&c.
\end{array}\right\}$$

$$p = \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}}, \&c.$$

The Terms being diſpos'd in order, the firſt Term of the Series $3a^2p$ will be x^3, which will make the firſt Term of p to be $\frac{x^3}{3a^2}$. This will make the firſt Term of p^2 to be $\frac{x^6}{9a^4}$. And this will make the firſt Term of $3ap^2$ to be $\frac{x^6}{3a^3}$, which with a contrary Sign muſt be the ſecond Term of $3a^2p$, and therefore the ſecond Term of p will be $-\frac{x^6}{9a^5}$. Then (by ſquaring) the ſecond Term of $3ap^2$ will be $-\frac{2x^9}{9a^6}$, and (by cubing) the firſt Term of p^3 will be $\frac{x^9}{27a^6}$. Theſe being collected make $-\frac{5x^9}{27a^6}$, which with a contrary Sign muſt be the third Term of $3a^2p$, and therefore the third Term of p will be $+\frac{5x^9}{81a^8}$. Then by ſquaring, the third Term of $3ap^2$ will be $\frac{13x^{12}}{81a^9}$, and by cubing, the ſecond Term of p^3 will be $-\frac{x^{12}}{27a^9}$, which being collected will make $\frac{10x^{12}}{81a^9}$; and therefore the fourth Term of $3a^2p$ will be $-\frac{10x^{12}}{81a^9}$, and the fourth Term of p will be $-\frac{10x^{12}}{243a^{11}}$. And ſo on.

And thus may the Roots of all pure Equations be extracted, but in a more direct and ſimple manner by the foregoing Theorems. All that is here intended, is, to prepare the way for the Reſolution of affected Equations, both in Numbers and Species, as alſo of Fluxional Equations, in which this Method will be found to be of very extenſive uſe. And firſt we ſhall proceed with our Author to the Solution of numerical affected Equations.

SECT. III. *The Reſolution of Numeral Affected Equations.*

19. NOW as to the Reſolution of affected Equations, and firſt in Numbers; our Author very juſtly complains, that before his time the *exegeſis numeroſa*, or the Doctrine of the Solution of affected Equations in Numbers, was very intricate, defective, and inartificial. What had been done by *Vieta*, *Harriot*, and *Oughtred* in this matter, tho' very laudable Attempts for the time, yet however was extremely perplex'd and operoſe. So that he had good reaſon to reject their Methods, eſpecially as he has ſubſtituted a much better in their room. They affected too great accuracy in purſuing

exact Roots, which led them into tedious perplexities; but he knew very well, that legitimate Approximations would proceed much more regularly and expeditiouſly, and would anſwer the ſame intention much better.

20, 21, 22. His Method may be eaſily apprehended from this one Inſtance, as it is contain'd in his Diagram, and the Explanation of it. Yet for farther Illuſtration I ſhall venture to give a ſhort *rationale* of it. When a Numeral Equation is propos'd to be reſolved, he takes as near an Approximation to the Root as can be readily and conveniently obtain'd. And this may always be had, either by the known Method of Limits, or by a Linear or Mechanical Conſtruction, or by a few eaſy trials and ſuppoſitions. If this be greater or leſs than the Root, the Exceſs or Defect, indifferently call'd the Supplement, may be repreſented by p, and the aſſumed Approximation, together with this Supplement, are to be ſubſtituted in the given Equation inſtead of the Root. By this means, (expunging what will be ſuperfluous,) a Supplemental Equation will be form'd, whoſe Root is now p, which will conſiſt of the Powers of the aſſumed Approximation orderly deſcending, involved with the Powers of the Supplement regularly aſcending, on both which accounts the Terms will be continually decreaſing, in a decuple ratio or faſter, if the aſſumed Approximation be ſuppos'd to be at leaſt ten times greater than the Supplement. Therefore to find a new Approximation, which ſhall nearly exhauſt the Supplement p, it will be ſufficient to retain only the two firſt Terms of this Equation, and to ſeek the Value of p from the reſulting ſimple Equation. [Or ſometimes the three firſt Terms may be retain'd, and the Value of p may be more accurately found from the reſulting Quadratick Equation; &c.] This new Approximation, together with a new Supplement q, muſt be ſubſtituted inſtead of p in this laſt ſupplemental Equation, in order to form a ſecond, whoſe Root will be q. And the ſame things may be obſerved of this ſecond ſupplemental Equation as of the firſt; and its Root, or an Approximation to it, may be diſcover'd after the ſame manner. And thus the Root of the given Equation may be proſecuted as far as we pleaſe, by finding new ſupplemental Equations, the Root of every one of which will be a correction to the preceding Supplement.

So in the preſent Example $y^3 - 2y - 5 = 0$, 'tis eaſy to perceive, that $y = 2$ *ferè*; for $2 \times 2 \times 2 - 2 \times 2 = 4$, which ſhould make 5. Therefore let p be the Supplement of the Root, and it will be $y = 2 + p$, and therefore by ſubſtitution $-1 + 10p + 6p^2 + p^3 = 0$. As p is here ſuppos'd to be much leſs than the Approximation 2, by this ſubſtitution an Equation will be form'd, in which the Terms will gradually decreaſe, and ſo much the faſter, *cæteris paribus*, as 2 is greater than p. So taking the two firſt Terms, $-1+10p=0$, *ferè*, or $p=\frac{1}{10}$ *ferè*; or aſſuming a ſecond Supplement q, 'tis $p=\frac{1}{10}+q$ accurately. This being ſubſtituted for p in the laſt Equation, it becomes $0{,}61+11{,}23q+6{,}3q^2+q^3=0$, which is a new Supplemental Equation, in which all the Terms are farther depreſs'd, and in which the Supplement q will be much leſs than the former Supplement p. Therefore it is $0{,}61+11{,}23q=0$, *ferè*, or $q=-\frac{0{,}61}{11{,}23}$ *ferè*, or $q=-0{,}0054+r$ *accuratè*, by aſſuming r for the third Supplement. This being ſubſtituted will give $0{,}00054155+11{,}162r$, &c. $=0$, and therefore $r=-\frac{0{,}00054155}{11{,}162}$ $=-0{,}000048 52$, &c. So that at laſt $y=2+p=$ &c. or $y=$ $2{,}09455148$, &c.

And thus our Author's Method proceeds, for finding the Roots of affected Equations in Numbers. Long after this was wrote, Mr. *Raphſon* publiſh'd his *Analyſis Æquationum univerſalis*, containing a Method for the Solution of Numeral Equations, not very much different from this of our Author, as may appear by the following Compariſon.

To find the Root of the Equation $y^3-2y=5$, Mr. *Raphſon* would proceed thus. His firſt Approximation he calls g, which he takes as near the true Root as he can, and makes the Supplement x, ſo that he has $y=g+x$. Then by Subſtitution

$$\begin{array}{l} g^3+3g^2x+3gx^2+x^3=5, \\ -2g-2 \end{array}$$

or if $g=2$, 'tis $10x+6x^2+x^3=1$, to determine the Supplement x. This being ſuppoſed ſmall, its Powers may be rejected, and therefore $10x=1$, or $x=0{,}1$ nearly. This added to g or 2, makes a new $g=2{,}1$, and x being ſtill the Supplement, 'tis $y=$ $2{,}1+x$, which being ſubſtituted in the original Equation y^3-2y $=5$, produces $11{,}23x+6{,}3x^2+x^3=-0{,}61$, to determine the new Supplement x. He rejects the Powers of x, and thence derives $x=\frac{-0{,}061}{11{,}23}=-0{,}0054$, and conſequently $y=2{,}0946$, which not being exact, becauſe the Powers of x were rejected, he makes the Supplement again to be x, ſo that $y=2{,}0946+x$, which being ſubſtituted in the Original Equation, gives $11{,}162x+$ &c. $=$ $-0{,}00054155$. Therefore to find the third Supplement x, he has $x=\frac{-0{,}00054155}{11{,}162}=-0{,}00004852$, ſo that $y=2{,}0946+x=$ $2{,}09455148$, &c. and ſo on.

By this Proceſs we may ſee how nearly theſe two Methods agree, and wherein they differ. For the difference is only this, that our Author conſtantly proſecutes the Reſidual or Supplemental Equations, to find the firſt, ſecond, third, *&c.* Supplements to the Root: But Mr. *Raphſon* continually corrects the Root itſelf from the ſame ſupplemental Equations, which are formed by ſubſtituting the corrected Roots in the Original Equation. And the Rate of Convergency will be the ſame in both.

In imitation of theſe Methods, we may thus proſecute this Inquiry after a very general manner. Let the given Equation to be reſolved be in this form $ay^m + by^{m-1} + cy^{m-2} + dy^{m-3}$, &c. $= 0$, in which ſuppoſe P to be any near Approximation to the Root y, and the little Supplement to be p. Then is $y = P + p$. Now from what is ſhewn before, concerning the raiſing of Powers and extracting Roots, it will follow that $y^m = \overline{P+p}|^m = P^m + mP^{m-1}p$, &c. or that theſe will be the two firſt Terms of y^m; and all the reſt, being multiply'd into the Powers of p, may be rejected. And for the ſame reaſon $y^{m-1} = P^{m-1} + \overline{m-1}P^{m-2}p$, &c. $y^{m-2} = P^{m-2} + \overline{m-2}P^{m-3}p$, &c. and ſo of all the reſt. Therefore theſe being ſubſtituted into the Equation, it will be

$$\left.\begin{array}{ll} aP^m & + maP^{m-1}p\text{, \&c.} \\ + bP^{m-1} & + \overline{m-1}bP^{m-2}p\text{, \&c.} \\ + cP^{m-2} & + \overline{m-2}cP^{m-3}p\text{, \&c.} \\ + dP^{m-3} & + \overline{m-3}dP^{m-4}p\text{, \&c.} \\ \text{\&c.} & \text{\&c.} \end{array}\right\} = 0\text{; Or dividing by } P^m\text{,}$$

$a + bP^{-1} + cP^{-2} + dP^{-3}$, &c. $+ maP^{-1}p + \overline{m-1}bP^{-2}p + \overline{m-2}cP^{-3}p + \overline{m-3}dP^{-4}p$, &c. $= 0$. From whence taking the Value of p, we ſhall have $p = -\dfrac{a + bP^{-1} + cP^{-2} + dP^{-3}\text{, \&c.}}{maP^{-1} + \overline{m-1}bP^{-2} + \overline{m-2}cP^{-3} + \overline{m-3}dP^{-4}\text{, \&c.}}$, and conſequently $y = (P + p = P - \dfrac{a + bP^{-1} + cP^{-2} + dP^{-3}\text{, \&c}}{maP^{-1} + \overline{m-1}bP^{-2} + \overline{m-2}cP^{-3} + \overline{m-3}dP^{-4}\text{, \&c.}} =)$ $\dfrac{\overline{m-1}a + \overline{m-2}bP^{-1} + \overline{m-3}cP^{-2} + \overline{m-4}dP^{-3}\text{, \&c.}}{maP^{-1} + \overline{m-1}bP^{-2} + \overline{m-2}cP^{-3} + \overline{m-3}dP^{-3}\text{, \&c.}}$

To reduce this to a more commodious form, make $P = \frac{A}{B}$, whence $P^{-1} = A^{-1}B$, $P^{-2} = A^{-2}B^2$, &c. which being ſubſtituted, and alſo multiplying the Numerator and Denominator by A^m, it will be $y = \dfrac{\overline{m-1}aA^m + \overline{m-2}bA^{m-1}B + \overline{m-3}cA^{m-2}B^2 + \overline{m-4}dA^{m-3}B^3\text{, \&c.}}{maA^{m-1}B + \overline{m-1}bA^{m-2}B^2 + \overline{m-2}cA^{m-3}B^3 + \overline{m-3}dA^{m-4}B^4\text{, \&c.}}$ which will be a nearer Approach to the Root y, than $\frac{A}{B}$, or P, and ſo much the nearer as $\frac{A}{B}$ is near the Root. And hence we may derive a very convenient and general Theorem for the Extraction of the Roots of Numeral Equations, whether pure or affected, which will be this.

Let the general Equation $ay^m + by^{m-1} + cy^{m-2} + dy^{m-3}$, &c. $= 0$ be propoſed to be ſolved; if the Fraction $\frac{A}{B}$ be aſſumed as near the Root y as conveniently may be, the Fraction $\frac{\overline{m-1}aA^m + \overline{m-2}bA^{m-1}B + \overline{m-3}cA^{m-2}B^2 + \overline{m-4}dA^{m-3}B^3, \&c.}{m2A^{m-1}B + \overline{m-1}bA^{m-2}B^2 + \overline{m-2}cA^{m-3}B^3 + \overline{m-3}dA^{m-4}B^4, \&c.}$ will be ſtill a nearer Approximation to the Root. And this Fraction, when computed, may be uſed inſtead of the Fraction $\frac{A}{B}$, by which means a nearer Approximation may again be had; and ſo on, till we approach as near the true Root as we pleaſe.

This general Theorem may be conveniently reſolved into as many particular Theorems as we pleaſe. Thus in the Quadratick Equation $y^2 + by = c$, it will be $y = \frac{A^2 + cB^2}{\overline{2A + bB} \times B}$, *ferè*. In the Cubick Equation $y^3 + by^2 + cy = d$, it will be $y = \frac{\overline{2A + bB} \times A^2 + dB^3}{\overline{3A^2 + 2bAB + cB^2} \times B}$, *ferè*. In the Biquadratick Equation $y^4 + by^3 + cy^2 + dy = e$, it will be $y = \frac{\overline{3A^2 + 2bAB + cB^2} \times A^2 + eB^4}{\overline{4A^3 + 3bA^2B + 2cAB^2 + aB^3} \times B}$, *ferè*. And the like of higher Equations.

For an Example of the Solution of a Quadratick Equation, let it be propoſed to extract the Square-root of 12, or let us find the value of y in this Equation $y^2 * = 12$. Then by comparing with the general *formula*, we ſhall have $b = 0$, and $c = 12$. And taking 3 for the firſt approach to the Root, or making $\frac{A}{B} = \frac{3}{1}$, that is, $A = 3$ and $B = 1$, we ſhall have by Subſtitution $y = \frac{9+12}{6} = \frac{7}{2}$, for a nearer Approximation. Again, making $A = 7$ and $B = 2$, we ſhall have $y = \frac{49+48}{14 \times 2} = \frac{97}{28}$ for a nearer Approximation. Again, making $A = 97$ and $B = 28$, we ſhall have $y = \frac{\overline{97}|^2 + 12 \times \overline{28}|^2}{194 \times 28} = \frac{18817}{5432}$ for a nearer Approximation. Again, making $A = 18817$ and $B = 5432$, we ſhall have $y = \frac{\overline{18817}|^2 + 12 \times \overline{5432}|^2}{37634 \times 5432} = \frac{708158977}{204427888}$ for a nearer Approximation. And if we go on in the ſame method, we may find as near an Approximation to the Root as we pleaſe.

This Approximation will be exhibited in a vulgar Fraction, which, if it be always kept to its loweſt Terms, will give the Root of the Equation in the ſhorteſt and ſimpleſt manner. That is, it will always be nearer the true Root than any other Fraction whatever, whoſe Numerator and Denominator are not much larger Numbers than its own. If by Diviſion we reduce this laſt Fraction to a Decimal, we ſhall have 3,4641016151377545 9 for the Square-root of 12, which exceeds the truth by leſs than an Unit in the laſt place.

For an Example of a Cubick Equation, we will take that of our Author $y^3 * - 2y = 5$, and therefore by Compariſon $b = 0$, $c = -2$, and $d = 5$. And taking 2 for the firſt Approach to the Root, or making $\frac{A}{B} = \frac{2}{1}$, that is, $A = 2$ and $B = 1$, we ſhall have by Subſtitution $y = \frac{16+5}{12-2} = \frac{21}{10}$ for a nearer Approach to the Root. Again, make $A = 21$ and $B = 10$, and then we ſhall have $y = \frac{9261 + 2500}{6615 - 1000} = \frac{11761}{5615}$ for a nearer Approximation. Again, make $A = 11761$ and $B = 5615$, and we ſhall have $y = \frac{2 \times \overline{11761}|^3 + 5 \times \overline{5615}|^3}{3 \times \overline{11761}|^2 \times 5615 - 2 \times \overline{5615}|^3} = \frac{4138744325037}{1975957316495}$ for a nearer Approximation. And ſo we might proceed to find as near an Approximation as we think fit. And when we have computed the Root near enough in a Vulgar Fraction, we may then (if we pleaſe) reduce it to a Decimal by Diviſion. Thus in the preſent Example we ſhall have $y = 2,09455148170 1$, &c. And after the ſame manner we may find the Roots of all other numeral affected Equations, of whatever degree they may be.

SECT. IV. *The Reſolution of Specious Equations by infinite Series; and firſt for determining the forms of the Series, and their initial Approximations.*

23, 24. FROM the Reſolution of numeral affected Equations, our Author proceeds to find the Roots of Literal, Specious, or Algebraical Equations alſo, which Roots are to be exhibited by an infinite converging Series, conſiſting of ſimple Terms. Or they are to be expreſs'd by Numbers belonging to a general Arithmetical Scale, as has been explain'd before, of which the Root is denoted by x or z. The aſſigning or chuſing this Root is what he means here, by diſtinguiſhing one of the literal Coefficients from the reſt, if there are ſeveral. And this is done by ordering or diſpoſing the Terms of the given Equation, according to the Dimenſions of that Letter or Coefficient. It is therefore convenient to chuſe ſuch a Root of the Scale, (when choice is allow'd,) as that the Series may converge as faſt as may be. If it be the leaſt, or a Fraction leſs than Unity, its aſcending Powers muſt be in the Numerators of the Terms. If it be the greateſt quantity, then its aſcending Powers muſt be in the Denominators, to make the Series duly converge. If it be very near a given quantity, then that quantity may be conveniently made the firſt Approximation, and that ſmall difference, or Supplement, may be made the Root of the Scale, or the converging quantity. The Examples will make this plain.

25, 26. The Equation to be reſolved, for conveniency-ſake, ſhould always be reduced to the ſimpleſt form it can be, before its Reſolution be attempted; for this will always give the leaſt trouble. But all the Reductions mention'd by the Author, and of which he gives us Examples, are not always neceſſary, tho' they may be often convenient. The Method is general, and will find the Roots of Equations involving fractional or negative Powers, as well as of other Equations, as will plainly appear hereafter.

27, 28. When a literal Equation is given to be reſolved, in diſtinguiſhing or aſſigning a proper quantity, by which its Root is to converge, the Author before has made three caſes or varieties; all which, for the ſake of uniformity, he here reduces to one. For becauſe the Series muſt neceſſarily converge, that quantity muſt be as ſmall as poſſible, in reſpect of the other quantities, that its aſcending Powers may continually diminiſh. If it be thought proper to chuſe the greateſt quantity, inſtead of that its Reciprocal muſt be introduced, which will bring it to the foregoing caſe. And if it approach near to a given quantity, then their ſmall difference may be introduced into the Equation, which again will bring it to the firſt caſe. So that we need only purſue that caſe, becauſe the Equation is always ſuppos'd to be reduced to it.

But before we can conveniently explain our Author's Rule, for finding the firſt Term of the Series in any Equation, we muſt conſider the nature of thoſe Numbers, or Expreſſions, to which theſe literal Equations are reduced, whoſe Roots are required; and in this Inquiry we ſhall be much aſſiſted by what has been already diſcourſed of Arithmetical Scales. In affected Equations that were purely numeral, the Solution of which was juſt now taught, the ſeveral Powers of the Root were orderly diſpoſed, according to a ſingle or ſimple Arithmetical Scale, which proceeded only *in longum*, and was there ſufficient for their Solution. But we muſt enlarge our views in theſe literal affected Equations, in which are found, not only the Powers of the Root to be extracted, but alſo the Powers of the Root of the Scale, or of the converging quantity, by which the Series for the Root of the Equation is to be form'd; on account of each of which circumſtances the Terms of the Equation are to be regularly diſpoſed, and therefore are to conſtitute a double or combined Arithmetical Scale, which muſt proceed both ways, *in latum* as well as *in longum*, as it were in a Table. For the Powers of the Root to be extracted, ſuppoſe y, are to be diſpoſed *in longum*, ſo as that their Indices may conſtitute an Arithmetical Progreſſion, and the vacancies, if any, may be ſupply'd by the Mark $*$. Alſo the Indices of the Powers of the Root, by which the Series is to converge, ſuppoſe x, are to be diſpoſed *in latum*, ſo as to conſtitute an Arithmetical Progreſſion, and the vacancies may likewiſe be fill'd up by the ſame Mark $*$, when it ſhall be thought neceſſary. And both theſe together will make a combined or double Arithmetical Scale. Thus if the Equation $y^6 - 5xy^5 + \frac{x^3}{a}y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$, were given, to find the Root y, the Terms may be thus diſpoſed:

$$\begin{array}{c|ccccccc} & y^6 & y^5 & y^4 & y^3 & y^2 & y^1 & y^0 \\ \hline x^0 & y^6 & * & * & * & * & * & * \\ x & * & -5xy^5 & * & * & * & * & * \\ x^2 & * & * & * & * & -7a^2x^2y^2 & * & * \\ x^3 & * & * & +\frac{x^3}{a}y^4 & * & * & * & +6a^3x^3 \\ x^4 & * & * & * & * & * & * & +b^2x^4 \end{array} \left.\vphantom{\begin{array}{c}x\\x\\x\\x\\x\end{array}}\right\} = 0.$$

Alſo the Equation $y^5 - by^2 + 9bx^2 - x^3 = 0$ ſhould be thus diſpoſed, in order to its Solution:

$$\left.\begin{array}{cccccc} y^5 & * & * & -by^2 & * & * \\ & & & & & * \\ & & & & & +9bx^2 \\ & & & & & -x^3 \end{array}\right\} = 0.$$

And the Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$ thus:

$$\left.\begin{array}{cccc} y^3 & * & +a^2y & -2a^3 \\ & & +axy & * \\ & & & * \\ & & & -x^3 \end{array}\right\} = 0.$$

And the Equation $x^2y^5 - 3c^4xy^2 - c^5x^2 + c^7 = 0$ thus:

$$\left.\begin{array}{cccccc} * & * & * & * & * & +c^7 \\ * & * & * & -3c^4xy^2 & * & * \\ x^2y^5 & * & * & * & * & -c^5x^2 \end{array}\right\} = 0.$$

And the like of all other Equations.

When the Terms of the Equation are thus regularly diſpoſ'd, it is then ready for Solution; to which the following Speculation will be a farther preparation.

29. This ingenious contrivance of our Author, (which we may call Tabulating the Equation,) for finding the firſt Term of the Root, (which may indeed be extended to the finding all the Terms, or the form of the Series, or of all the Series that may be derived from the given Equation,) cannot be too much admired, or too carefully inquired into: The reaſon and foundation of which may be thus generally explain'd from the following Table, of which the Conſtruction is thus.

$-2a+6b$	$-a+6b$	$+6b$	$a+6b$	$2a+6b$	$3a+6b$	$4a+6b$	$5a+6b$	$6a+6b$	$7a+6b$
$-2a+5b$	$-a+5b$	$+5b$	$a+5b$	$2a+5b$	$3a+5b$	$4a+5b$	$5a+5b$	$6a+5b$	$7a+5b$
$-2a+4b$	$-a+4b$	$+4b$	$a+4b$	$2a+4b$	$3a+4b$	$4a+4b$	$5a+4b$	$6a+4b$	$7a+4b$
$-2a+3b$	$-a+3b$	$+3b$	$a+3b$	$2a+3b$	$3a+3b$	$4a+3b$	$5a+3b$	$6a+3b$	$7a+3b$
$-2a+2b$	$-a+2b$	$+2b$	$a+2b$	$2a+2b$	$3a+2b$	$4a+2b$	$5a+2b$	$6a+2b$	$7a+2b$
$-2a+b$	$-a+b$	$+b$	$a+b$	$2a+b$	$3a+b$	$4a+b$	$5a+b$	$6a+b$	$7a+b$
$-2a$	$-a$	0	a	$2a$	$3a$	$4a$	$5a$	$6a$	$7a$
$-2a-b$	$-a-b$	$-b$	$a-b$	$2a-b$	$3a-b$	$4a-b$	$5a-b$	$6a-b$	$7a-b$
$-2a-2b$	$-a-2b$	$-2b$	$a-2b$	$2a-2b$	$3a-2b$	$4a-2b$	$5a-2b$	$6a-2b$	$7a-2b$
$-2a-3b$	$-a-3b$	$-3b$	$a-3b$	$2a-3b$	$3a-3b$	$4a-3b$	$5a-3b$	$6a-3b$	$7a-3b$

In a Plane draw any number of Lines, parallel and equidiſtant, and others at right Angles to them, ſo as to divide the whole Space, as far as is neceſſary, into little equal Parallelograms. Aſſume any one of theſe, in which write the Term 0, and the Terms a, $2a$, $3a$, $4a$, &c. in the ſucceeding Parallelograms to the right hand, as alſo the Terms $-a$, $-2a$, $-3a$, &c. to the left hand. Over the Term 0, in the ſame Column, write the Terms b, $2b$, $3b$, $4b$, &c. ſucceſſively, and the Terms $-b$, $-2b$, $-3b$, &c. underneath. And theſe we may call primary Terms. Now to inſert its proper Term in any other aſſign'd Parallelogram, add the two primary Terms together that ſtand over-againſt it each way, and write the Sum in the given Parallelogram. And thus all the Parallelograms being fill'd, as far as there is occaſion every way, the whole Space will become a Table, which may be called *a combined Arithmetical Progreſſion in plano*, compoſed of the two general Numbers a and b, of which theſe following will be the chief properties.

Any Row of Terms, parallel to the primary Series $0, a, 2a, 3a,$ &c. will be an Arithmetical Progreſſion, whoſe common Difference is a; and it may be any ſuch Progreſſion at pleaſure. Any Row or Column parallel to the primary Series $0, b, 2b, 3b,$ &c. will be an Arithmetical Progreſſion, whoſe common difference is b; and it may be any ſuch Progreſſion. If a ſtrait Ruler be laid on the Table, the Edge of which ſhall paſs thro' the Centers of any two Parallelograms whatever; all the Terms of the Parallelograms, whoſe Centers ſhall at the ſame time touch the Edge of the Ruler, will conſtitute an Arithmetical Progreſſion, whoſe common difference will conſiſt of two parts, the firſt of which will be ſome Multiple of a, and the other a Multiple of b. If this Progreſſion be ſuppos'd to proceed *inferiora verſus*, or from the upper Term or Parallelogram towards the lower; each part of the common difference may be ſeparately found, by ſubtracting the primary Term belonging to the lower, from the primary Term belonging to the upper Parallelogram. If this common difference, when found, be made equal to nothing, and thereby the Relation of a and b be determined; the Progreſſion degenerates into a Rank of Equals, or (if you pleaſe) it becomes an Arithmetical Progreſſion, whoſe common difference is infinitely little. In which caſe, if the Ruler be moved by a parallel motion, all the Terms of the Parallelograms, whoſe Centers ſhall at the ſame time be found to touch the Edge of the Ruler, ſhall be equal to each other. And if the motion of the Ruler be continued, ſuch Terms as at equal diſtances from the firſt ſituation are ſucceſſively found to touch the Ruler, ſhall form an Arithmetical Progreſſion. Laſtly, to come nearer to the caſe in hand, if any number of theſe Parallelograms be mark'd out and diſtinguiſh'd from the reſt, or aſſign'd promiſcuouſly and at pleaſure, through whoſe Centers, as before, the Edge of the Ruler ſhall ſucceſſively paſs in its parallel motion, beginning from any two (or more) initial or external Parallelograms, whoſe Terms are made equal; an Arithmetical Progreſſion may be found, which ſhall comprehend and take in all thoſe promiſcuous Terms, without any regard had to the Terms that are to be omitted. Theſe are ſome of the properties of this Table, or of a combined Arithmetical Progreſſion *in plano*, by which we may eaſily underſtand our Author's expedient, of Tabulating the given Equation, and may derive the neceſſary Conſequences from it.

For when the Root y is to be extracted out of a given Equation, conſiſting of the Powers of y and x any how combined together promiſcuouſly, with other known quantities, of which x is to be the Root of the Scale, (or Series,) as explain'd before; ſuch a value of y is to be found, as when ſubſtituted in the Equation inſtead of y, the whole ſhall be deſtroy'd, and become equal to nothing. And firſt the initial Term of the Series, or the firſt Approximation, is to be found, which in all caſes may be Analytically repreſented by Ax^m; or we may always put $y = Ax^m$, &c. So that we ſhall have $y^2 = A^2x^{2m}$, &c. $y^3 = A^3x^{3m}$, &c. $y^4 = A^4x^{4m}$, &c. And ſo of other Powers or Roots. Theſe when ſubſtituted in the Equation, and by that means compounded with the ſeveral Powers of x (or z) already found there, will form ſuch a combined Arithmetical Progreſſion *in plano* as is above deſcribed, or which may be reduced to ſuch, by making $a = m$ and $b = 1$. Theſe Terms therefore, according to the nature of the Equation, will be promiſcuouſly diſperſed in the Table; but the vacancies may always be conceived to be ſupply'd, and then it will have the properties before mention'd. That is, the Ruler being apply'd to two (or perhaps more) initial or external Terms, (for if they were not external, they could not be at the beginning of an Arithmetical Progreſſion, as is neceſſarily required,) and thoſe Terms being made equal, the general Index m will thereby be determined, and the general Coefficient A will alſo be known. If the external Terms made choice of are the loweſt in the Table, which is the caſe our Author purſues, the Powers of x will proceed by increaſing. But the higheſt may be choſen, and then a Series will be found, in which the Powers of x will proceed by decreaſing. And there may be other caſes of external Terms, each of which will commonly afford a Series. The initial Index being thus found, the other compound Indices belonging to the Equation will be known alſo, and an Arithmetical Progreſſion may be found, in which they are all comprehended, and conſequently the form of the Series will be known.

Or inſtead of Tabulating the Indices of the Equation, as above, it will be the ſame thing in effect, if we reduce the Terms themſelves to the form of a combined Arithmetical Progreſſion, as was ſhewn before. But then due care muſt be taken, that the Terms may be rightly placed at equal diſtances; otherwiſe the Ruler cannot be actually apply'd, to diſcover the Progreſſions of the Indices, as may be done in the Parallelogram.

For the ſake of greater perſpicuity, we will reduce our general Table, or combined Arithmetical Progreſſion *in plano*, to the particular caſe, in which $a = m$ and $b = 1$; which will then appear thus:

$-2m+6$	$-m+6$	$+6$	$m+6$	$2m+6$	$3m+6$	$4m+6$	$5m+6$	$6m+6$	$7m+6$
$-2m+5$	$-m+5$	$+5$	$m+5$	$2m+5$	$3m+5$	$4m+5$	$5m+5$	$6m+5$	$7m+5$
$-2m+4$	$-m+4$	$+4$	$m+4$	$2m+4$	$3m+4$	$4m+4$	$5m+4$	$6m+4$	$7m+4$
$-2m+3$	$-m+3$	$+3$	$m+3$	$2m+3$	$3m+3$	$4m+3$	$5m+3$	$6m+3$	$7m+3$
$-2m+2$	$-m+2$	$+2$	$m+2$	$2m+2$	$3m+2$	$4m+2$	$5m+2$	$6m+2$	$7m+2$
$-2m+1$	$-m+1$	$+1$	$m+1$	$2m+1$	$3m+1$	$4m+1$	$5m+1$	$6m+1$	$7m+1$
$-2m$	$-m$	0	m	$2m$	$3m$	$4m$	$5m$	$6m$	$7m$
$-2m-1$	$-m-1$	-1	$m-1$	$2m-1$	$3m-1$	$4m-1$	$5m-1$	$6m-1$	$7m-1$
$-2m-2$	$-m-2$	-2	$m-2$	$2m-2$	$3m-2$	$4m-2$	$5m-2$	$6m-2$	$7m-2$
$-2m-3$	$-m-3$	-3	$m-3$	$2m-3$	$3m-3$	$4m-3$	$5m-3$	$6m-3$	$7m-3$

Now the chief properties of this Table, ſubſervient to the preſent purpoſe, will be theſe. If any Parallelogram be ſelected, and another any how below it towards the right hand, and if their included Numbers be made equal, by determining the general Number m, which in this caſe will always be affirmative; alſo if the Edge of the Ruler be apply'd to the Centers of theſe two Parallelograms; all the Numbers of the other Parallelograms, whoſe Centers at the ſame time touch the Ruler, will likewiſe be equal to each other. Thus if the Parallelogram denoted by $m+4$ be ſelected, as alſo the Parallelogram $3m+2$; and if we make $m+4=3m+2$; we ſhall have $m=1$. Alſo the Parallelograms $-m+6$, $m+4$, $3m+2$, $5m$, $7m-2$, &c. will at the ſame time be found to touch the Edge of the Ruler, every one of which will make 5; when $m=1$.

And the ſame things will obtain if any Parallelogram be ſelected, and another any how below it towards the left-hand, if their included Numbers be made equal, by determining the general Number m, which in this caſe will be always negative. Thus if the Parallelogram denoted by $5m+4$ be ſelected, as alſo the Parallelogram $4m+2$; and if we make $5m+4=4m+2$, we ſhall have $m=-2$. Alſo the Parallelograms $6m+6$, $5m+4$, $4m+2$, $3m$, $2m-2$, &c.

will be found at the ſame time to touch the Ruler, every one of which will make -6, when $m = -2$.

The ſame things remaining as before, if from the firſt ſituation of the Ruler it ſhall move towards the right-hand by a parallel motion, it will continually arrive at greater and greater Numbers, which at equal diſtances will form an aſcending Arithmetical Progreſſion. Thus if the two firſt ſelected Parallelograms be $2m - 1 = 5m - 3$, whence $m = \frac{2}{3}$, the Numbers in all the correſponding Parallelograms will be $\frac{1}{3}$. Then if the Ruler moves towards the right-hand, into the parallel ſituation $3m + 1$, $6m - 1$, &c. theſe Numbers will each be 3. If it moves forwards to the ſame diſtance, it will arrive at $4m + 3$, $7m + 1$, &c. which will each be $5\frac{2}{3}$. If it moves forward again to the ſame diſtance, it will arrive at $5m + 5$, $8m + 3$, &c. which will each be $8\frac{1}{3}$. And ſo on. But the Numbers $\frac{1}{3}$, 3, $5\frac{2}{3}$, $8\frac{1}{3}$, &c. are in an Arithmetical Progreſſion whoſe common difference is $2\frac{2}{3}$. And the like, *mutatis mutandis*, in other circumſtances.

And hence it will follow *è contra*, that if from the firſt ſituation of the Ruler, it moves towards the left-hand by a parallel motion, it will continually arrive at leſſer and leſſer Numbers, which at equal diſtances will form a decreaſing Arithmetical Progreſſion.

But in the other ſituation of the Ruler, in which it inclines downwards towards the left-hand, if it be moved towards the right-hand by a parallel motion, it will continually arrive at greater and greater Numbers, which at equal diſtances will form an increaſing Arithmetical Progreſſion. Thus if the two firſt ſelected Numbers or Parallelograms be $8m + 1 = 5m - 1$, whence $m = -\frac{2}{3}$, and the Numbers in all the correſponding Parallelograms will be $-4\frac{1}{3}$. If the Ruler moves upwards into the parallel ſituation $5m + 2$, $2m$, &c. theſe Numbers will each be $-1\frac{1}{3}$. If it move on at the ſame diſtance, it will arrive at $2m + 3$, $-m + 1$, &c. which will each be $1\frac{2}{3}$. If it move forward again to the ſame diſtance, it will arrive at $-m + 4$, $-4m + 2$, &c. which will each be $4\frac{2}{3}$. And ſo on. But the Numbers $-4\frac{1}{3}$, $-1\frac{1}{3}$, $1\frac{2}{3}$, $4\frac{2}{3}$, &c. or $-\frac{13}{3}$, $-\frac{4}{3}$, $\frac{5}{3}$, $\frac{14}{3}$, &c. are in an increaſing Arithmetical Progreſſion, whoſe common difference is $\frac{9}{3}$, or 3.

And hence it will follow alſo, if in this laſt ſituation of the Ruler it moves the contrary way, or towards the left-hand, it will continually arrive at leſſer and leſſer Numbers, which at equal diſtances will form a decreaſing Arithmetical Progreſſion.

Now if out of this Table we ſhould take promiſcuouſly any number of Parallelograms, in their proper places, with their reſpective

Numbers included, neglecting all the reſt; we ſhould form ſome certain Figure, ſuch as this, of which theſe would be the properties.

		$3m+5$		$5m+5$	
$m+3$			$4m+3$		$6m+3$
	$2m+1$			$5m+1$	

The Ruler being apply'd to any two (or perhaps more) of the Parallelograms which are in the Ambit or Perimeter of the Figure, that is, to two of the external Parallelograms, and their Numbers being made equal, by determining the general Number m; if the Ruler paſſes over all the reſt of the Parallelograms by a parallel motion, thoſe Numbers which at the ſame time come to the Edge of the Ruler will be equal, and thoſe that come to it ſucceſſively will form an Arithmetical Progreſſion, if the Terms ſhould lie at equal diſtances; or at leaſt they may be reduced to ſuch, by ſupplying any Terms that may happen to be wanting.

Thus if the Ruler ſhould be apply'd to the two uppermoſt and external Parallelograms, which include the Numbers $3m+5$ and $5m+5$, and if they be made equal, we ſhall have $m=0$, ſo that each of theſe Numbers will be 5. The next Numbers that the Ruler will arrive at will be $m+3$, $4m+3$, $6m+3$, of which each will be 3. The laſt are $2m+1$, $5m+1$, of which each is 1. So that here $m=0$, and the Numbers ariſing are 5, 3, 1, which form a decreaſing Arithmetical Progreſſion, the common difference of which is 2. And if there had been more Parallelograms, any how diſpoſed, their Numbers would have been comprehended by this Arithmetical Progreſſion, or at leaſt it might have been interpolated with other Terms, ſo as to comprehend them all, however promiſcuouſly and irregularly they might have been taken.

Thus ſecondly, if the Ruler be apply'd to the two external Parallelograms $5m+5$ and $6m+3$, and if theſe Numbers be made equal, we ſhall have $m=2$, and the Numbers themſelves will be each 15. The three next Numbers which the Ruler will arrive at will be each 11, and the two laſt will be each 5. But the Numbers 15, 11, 5, will be comprehended in the decreaſing Arithmetical Progreſſion 15, 13, 11, 9, 7, 5, whoſe common difference is 2.

Thirdly, if the Ruler be apply'd to the two external Parallelograms $6m + 3$ and $5m + 1$, and if theſe Numbers be made equal, we ſhall have $m = -2$, and the Numbers will be each -9. The two next Numbers that the Ruler will arrive at will be each -5, the next will be -3, the next -1, and the laſt $+1$. All which will be comprehended in the aſcending Arithmetical Progreſſion -9, -7, -5, -3, -1, $+1$, whoſe common difference is 2.

Fourthly, if the Ruler be apply'd to the two loweſt and external Parallelograms $2m + 1$ and $5m + 1$, and if they be made equal, we ſhall have again $m = 0$, ſo that each of theſe Numbers will be 1. The next three Numbers that the Ruler will approach to, will each be 3, and the laſt 5. But the Numbers 1, 3, 5, will be comprehended in an aſcending Arithmetical Progreſſion, whoſe common difference is 2.

Fifthly, if the Ruler be apply'd to the two external Parallelograms $m + 3$ and $2m + 1$, and if theſe Numbers be made equal, we ſhall have $m = 2$, and the Numbers themſelves will be each 5. The three next Numbers that the Ruler will approach to will each be 11, and the two next will be each 15. But the Numbers 5, 11, 15, will be comprehended in the aſcending Arithmetical Progreſſion 5, 7, 9, 11, 13, 15, of which the common difference is 2.

Laſtly, if the Ruler be apply'd to the two external Parallelograms $3m + 5$ and $m + 3$, and if theſe Numbers be made equal, we ſhall have $m = -1$, and the Numbers themſelves will each be 2. The next Number to which the Ruler approaches will be 0, the two next are each -1, the next -3, the laſt -4. All which Numbers will be found in the deſcending Arithmetical Progreſſion 2, 1, 0, -1, -2, -3, -4, whoſe common difference is 1. And theſe ſix are all the poſſible caſes of external Terms.

Now to find the Arithmetical Progreſſion, in which all theſe reſulting Terms ſhall be comprehended; find their differences, and the greateſt common Diviſor of thoſe differences ſhall be the common difference of the Progreſſion. Thus in the fifth caſe before, the reſulting Numbers were 5, 11, 15, whoſe differences are 6, 4, and their greateſt common Diviſor is 2. Therefore 2 will be the common difference of the Arithmetical Progreſſion, which will include all the reſulting Numbers 5, 11, 15, without any ſuperfluous Terms. But the application of all this will be beſt apprehended from the Examples that are to follow.

30. We have before given the form of this Equation, $y^6 - 5xy^5 + \frac{x^3}{a}y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$, when the Terms are diſpoſed according to a double or combined Arithmetical Scale, in order to its Solution. Or obſerving the ſame diſpoſition of the Terms, they may be inſerted in their reſpective Parallelograms, as the Table requires. Or rather, it may be ſufficient to tabulate the ſeveral Indices of x only, when they are derived as follows. Let Ax^m repreſent the firſt Term of the Series to be form'd for y, as before, or let $y = Ax^m$, &c. Then by ſubſtituting this for y in the given Equation, we ſhall have $A^6x^{6m} - 5A^5x^{5m+1} + \frac{A^4}{a}x^{4m+3} - 7a^2A^2x^{2m+2} + 6a^3x^3 + b^2x^4$, &c. $= 0$. Theſe Indices of x, when ſelected from the general Table, with their reſpective Parallelograms, will ſtand thus:

4						
3				$4m+3$		
		$2m+2$				
					$5m+1$	
						$6m$

Here if we would have an aſcending Series for the Root y, we may apply the Ruler to the three external Terms 3, $2m+2$, $6m$, which being made equal to each other, will give $m = \frac{1}{2}$, and each of the Numbers will be 3. The Ruler in its parallel motion will next arrive at $5m+1$, or $3\frac{1}{2}$; then at 4; then at $4m+3$, or 5; which Numbers will be comprehended in the Arithmetical Progreſſion 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, whoſe common difference is $\frac{1}{2}$. This therefore will be the common difference of the Progreſſion of the Indices, in the Series to be derived for y. So that now we intirely know the form of the Series, which will reſult from this Caſe. For if A, B, C, D, &c. be put to repreſent the ſeveral Coefficients of the Series in order, and as the firſt Index m is found to be $\frac{1}{2}$, and the common difference of the aſcending Series is alſo $\frac{1}{2}$, we ſhall have here $y = Ax^{\frac{1}{2}} + Bx + Cx^{\frac{3}{2}} + Dx^2$, &c.

As to the Value of the firſt Coefficient A, this is found by putting the initial or external Terms of the Parallelogram equal to nothing.

4

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This here will give the Equation $A^6 - 7a^2A^2 + 6a^3 = 0$, which has theſe ſix Roots, $A = \pm\sqrt{a}$, $A = \pm\sqrt{2a}$, $A = \pm\sqrt{-3a}$, of which the two laſt are impoſſible, and to be rejected. Of the others any one may be taken for A, according as we would proſecute this or that Root of the Equation.

Now that this is a legitimate Method for finding the firſt Approximation Ax^m, may appear from conſidering, that when the Terms of the Equation are thus ranged, according to a double Arithmetical Scale, the initial or external Terms, (each Caſe in its turn,) become the moſt conſiderable of the Series, and the reſt continually decreaſe, or become of leſs and leſs value, according as they recede more and more from thoſe initial Terms. Conſequently they may be all rejected, as leaſt conſiderable, which will make thoſe initial or external Terms to be (nearly) equal to nothing; which Suppoſition gives the Value of A, or of Ax^m, for the firſt Approximation. And this Suppoſition is afterwards regularly purſued in the ſubſequent Operations, and proper Supplements are found, by means of which the remaining Terms of the Root are extracted.

We may try here likewiſe, if we can obtain a deſcending Series for the Root y, by applying the Ruler to the two external Terms $4m + 3$ and $6m$; which being made equal to each other, will give $m = \frac{3}{2}$, and hence each of the Numbers will be 9. The Ruler in its motion will next arrive at $5m + 1$, or $8\frac{1}{2}$. Then at $2m + 2$, or 5. Then at 4. And laſtly at 3. But theſe Numbers 9, $8\frac{1}{2}$, 5, 4, 3, will be comprehended in an Arithmetical Progreſſion, of which the common difference is $\frac{1}{2}$. So that the form of the Series here will be $y = Ax^{\frac{3}{2}} + Bx + Cx^{\frac{1}{2}} + Dx^0$, &c. But if we put the two external Terms equal to nothing, in order to obtain the firſt Approximation, we ſhall have $A^6 + \frac{A^4}{a} = 0$, or $A^2 + \frac{1}{a} = 0$, which will afford none but impoſſible Roots. So that we can have no initial Approximation from this ſuppoſition, and conſequently no Series.

But laſtly, to try the third and laſt caſe of external Parallelograms, we may apply the Ruler to 4 and $4m + 3$, which being made equal, will give $m = \frac{1}{4}$, and each of the Numbers will be 4. The next Number will be 3; the next $2m + 2$, or $2\frac{1}{2}$; the next $5m + 1$, or $2\frac{1}{4}$; the laſt will be $6m$, or $1\frac{1}{2}$. But the Numbers 4, 3, $2\frac{1}{2}$, $2\frac{1}{4}$, $1\frac{1}{2}$, will all be found in a decreaſing Arithmetical Progreſſion, whoſe common difference will be $\frac{1}{4}$. So that $Ax^{\frac{1}{4}} + Bx^0 + Cx^{-\frac{1}{4}} + Dx^{-\frac{1}{2}}$, &c. may repreſent the form of this Series, if the circumſtances of the Coefficients will allow of an Approximation from hence. But if we make the initial Terms equal to nothing, we ſhall have $\frac{A^4}{a} + b^4 = 0$, which will give none but impoſſible Roots. So that we can have no initial Approximation from hence, and conſequently no Series for the Root in this form.

31. The Equation $y^5 - by^2 + 9bx^2 - x^3 = 0$, when the Terms are diſpoſed according to a double Arithmetical Scale, will have the form as was ſhewn before; from whence it may be known, what caſes of external Terms there are to be try'd, and what will be the circumſtances of the ſeveral Series for the Root y, which may be derived from hence. Or otherwiſe more explicitely thus. Putting Ax^m for the firſt Term of the Series y, this Equation will become by Subſtitution $A^5x^{5m} - bA^2x^{2m} + 9bx^2 - x^3$, &c. $= 0$. So that if we take theſe Indices of x out of the general Table, they will ſtand as in the following Diagram.

3					
2					
		$2m$			$5m$

Now in order to have an aſcending Series for y, we may apply the Ruler to the two external Parallelograms 2 and $2m$, which therefore being made equal, will give $m = 1$, and each of the Numbers will be 2. The Ruler then in its parallel progreſs will firſt come to 3, and then to $5m$, or 5. But the Numbers 2, 3, 5, are all contain'd in an aſcending Arithmetical Progreſſion, whoſe common difference is 1. Therefore the form of the Series will here be $y = Ax + Bx^2 + Cx^3$, &c. And to determine the firſt Coefficient A, we ſhall have the Equation $-bA^2x^2 + 9bx^2 = 0$, or $A^2 = 9$, that is $A = \pm 3$. So that either $+3x$, or $-3x$ may be the initial Approximation, according as we intend to extract the affirmative or the negative Root.

We ſhall have another caſe of external Terms, and perhaps another aſcending Series for y, by applying the Ruler to the Parallelograms $2m$ and $5m$, which Numbers being made equal, will give $m = 0$. (For by the way, when we put $2m = 5m$, we are not at liberty to argue by Diviſion, that $2 = 5$, becauſe this would bring us to an abſurdity. And the laws of Argumentation require, that no Abſurdities muſt be admitted, but when they are inevitable, and when they are of uſe to ſhew the falſity of ſome Suppoſition. We ſhould therefore here argue by Subtraction, thus: Becauſe $5m = 2m$, then $5m - 2m = 0$, or $3m = 0$, and therefore $m = 0$. This Caution I thought the more neceſſary, becauſe I have obſerved ſome,

who

who would lay the blame of their own Abſurdities upon the Analytical Art. But theſe Abſurdities are not to be imputed to the Art, but rather to the unſkilfulneſs of the Artiſt, who thus abſurdly applies the Principles of his Art.) Having therefore $m = 0$, we ſhall alſo have the Numbers $2m = 5m = 0$. The Ruler in its parallel motion will next arrive at 2; and then at 3. But the Numbers 0, 2, 3, will be comprehended in the Arithmetical Progreſſion 0, 1, 2, 3, whoſe common difference is 1. Therefore $y = A + Bx + Cx^2$, &c. will be the form of this Series. Now from the exterior Terms $A^5 - bA^2 = 0$, or $A^3 = b$, or $A = b^{\frac{1}{3}}$, we ſhall have the firſt Term of the Series.

There is another caſe of external Terms to be try'd, which poſſibly may afford a deſcending Series for y. For applying the Ruler to the Parallelograms 3 and $5m$, and making theſe equal, we ſhall have $m = \frac{3}{5}$, and each of theſe Numbers will be 3. Then the Ruler will come to 2; and laſtly $2m$, or $\frac{6}{5}$. But the Numbers 3, 2, $1\frac{1}{5}$, will be comprehended in a deſcending Progreſſion, whoſe common difference is $\frac{1}{5}$. Therefore the form of the Series will be $y = Ax^{\frac{3}{5}} + Bx^{\frac{2}{5}} + Cx^{\frac{1}{5}} + D$, &c. And the external Terms $A^5x^3 - x^3 = 0$ will give $A = 1$ for the firſt Coefficient. Now as the two former caſes will each give a converging Series for y in this Equation, when x is leſs than Unity; ſo this caſe will afford us a Series when x is greater than Unity; which will converge ſo much the faſter, the greater x is ſuppoſed to be.

32. We have already ſeen the form of this Equation $y^3 + axy + aay - x^3 - 2a^3 = 0$, when the Terms are diſpoſed according to a double Arithmetical Scale. And if we take the fictitious quantity Ax^m to repreſent the firſt Approximation to the Root y, we ſhall have by ſubſtitution $A^3x^{3m} + aAx^{m+1} + a^2Ax^m - x^3 - 2a^3$, &c. $= 0$. Theſe Terms, or at leaſt theſe Indices of x, being ſelected out of the general Table, will appear thus.

3			
	$m+1$		
0	m		$3m$

Now to obtain an aſcending Series for the Root y, we may apply the Ruler to the three external Terms 0, m, $3m$, which being made equal, will give $m = 0$. Therefore theſe Numbers are each 0. In the next place the Ruler will come to $m + 1$, or 1; and laſtly to 3. But the Numbers 0, 1, 3, are contain'd in the Arithmetical Progreſſion 0, 1, 2, 3, whoſe common difference is 1. Therefore the form of the Root is $y = A + Bx + Cx^2 + Dx^3$, &c. Now if the Equation $A^3 + a^2A - 2a^3 = 0$, (which is derived from the initial Terms,) is divided by the factor $A^2 + aA + 2a^2$, it will give the Quotient $A - a = 0$, or $A = a$ for the initial Term of the Root y.

If we would alſo derive a deſcending Series for this Equation, we may apply the Ruler to the external Parallelograms 3, $3m$, which being made equal to each other, will give $m = 1$; alſo theſe Numbers will each be 3. Then the Ruler will approach to $m + 1$, or 2; then to m, or 1; laſtly to 0. But the Numbers 3, 2, 1, 0, are a decreaſing Arithmetical Progreſſion, of which the common difference is 1. So that the form of the Series will here be $y = Ax + B + Cx^{-1} + Dx^{-2}$, &c. And the Equation form'd by the external Terms will be $A^3x^3 - x^3 = 0$, or $A = 1$.

33. The form of the Equation $x^2y^5 - 3c^4xy^2 - c^5x^2 + c^7 = 0$, as expreſs'd by a combined Arithmetical Scale, we have already ſeen, which will eaſily ſhew us all the varieties of external Terms, with their other Circumſtances. But for farther illuſtration, putting Ax^m for the firſt Term of the Root y, we ſhall have by ſubſtitution $A^5x^{5m+2} - 3c^4A^2x^{2m+1} - c^5x^2 + c^7$, &c. $= 0$. Theſe Indices of x being tabulated, will ſtand thus.

2					$5m+2$
		$2m+1$			
0					

Now to have an aſcending Series, we muſt apply the Ruler to the two external Terms 0 and $5m + 2$, which being made equal, will give $m = -\frac{2}{5}$, and the two Numbers ariſing will be each 0. The next Number that the Ruler arrives at is $2m + 1$, or $\frac{1}{5}$; and the laſt is 2. But the Numbers 0, $\frac{1}{5}$, 2, will be found in an aſcending Arithmetical Progreſſion, whoſe common difference is $\frac{1}{5}$. Therefore $y = Ax^{-\frac{2}{5}} + Bx^{\frac{1}{5}} + C + Dx^{\frac{1}{5}}$, &c. will be the form of the Root. To determine the firſt Coefficient A, we ſhall have from the exterior Terms $A^5 + c^7 = 0$, which will give $A = -\sqrt[5]{c^7} = -c^{\frac{7}{5}}$. Therefore the firſt Term or Approximation to the Root will be $y = -\sqrt[5]{\frac{c^7}{x^2}}$, &c.

We may try if we can obtain a deſcending Series, by applying the Ruler to the two external Parallelograms, whoſe Numbers are 2 and $5m + 2$, which being made equal, will give $m = 0$, and theſe Numbers will each be 2. The Ruler will next arrive at $2m + 1$, or 1; and laſtly at 0. But the Numbers 2, 1, 0, form a deſcending Progreſſion, whoſe common difference is 1. So that the form of the Series will here be $y = A + Bx^{-1} + Cx^{-2}$, &c. And putting the

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initial Terms equal to nothing, as they ſtand in the Equation, we ſhall have $A^2x^2 - c^2x^2 = 0$, or $A = c$, for the firſt Approximation to the Root. And this Series will be accommodated to the caſe of Convergency, when x is greater than c; as the other Series is accommodated to the other caſe, when x is leſs than c.

34. If the propoſed Equation be $8z^6y^3 + az^6y^2 - 27a^9 = 0$, it may be thus reſolved without any preparation. When reduced to our form, it will ſtand thus, $\left.\begin{array}{llll} 8z^6y^3 + az^6y^2 & * & * & \\ & * & -27a^9 & \end{array}\right\} = 0$; and by putting $y = Az^m$, &c. it will become $\left.\begin{array}{llll} 8A^3z^{3m+6} + aA^2z^{2m+6} & * & * & \&c. \\ & & * & -27a^9 \end{array}\right\} = 0$.
The firſt caſe of external Terms will give $8A^3z^{3m+6} - 27a^9 = 0$, whence $3m + 6 = 0$, or $m = -2$. Theſe Indices or Numbers therefore will be each 0; and the other $2m + 6$ will be 2. But 0, 2, will be in an aſcending Arithmetical Progreſſion, of which the common difference is 2. So that the form of the Series will be $y = Az^{-2} + B + Cz^2 + Dz^4$, &c. And becauſe $8A^3 = 27a^9$, or $2A = 3a^3$, it will be $A = \frac{3}{2}a^3$. Therefore the firſt Term or Approximation to the Root will be $\frac{3a^3}{2z^2}$.

But another caſe of external Terms will give $aA^2z^{2m+6} - 27a^9 = 0$, whence $2m + 6 = 0$, or $m = -3$. Theſe Indices or Numbers therefore will be each 0; and the other $3m + 6$ will be -3. But 0, -3, will be found in a deſcending Arithmetical Progreſſion, whoſe common difference is 3. So that the form of the Series will be $y = Az^{-3} + Bz^{-6} + Cz^{-9}$, &c. And becauſe $aA^2 = 27a^9$, 'tis $A = \pm 3\surd 3 \times a^4$, for the firſt Coefficient.

Laſtly, there is another caſe of external Terms, which may poſſibly afford us a deſcending Series, by making $8A^3z^{3m+6} + aA^2z^{2m+6} = 0$; whence $m = 0$. And the Numbers will be each equal to 6; the other Number, or Index of z, is 0. But 6, 0, will be in a deſcending Arithmetical Progreſſion, of which the common difference is 6. Therefore the form of the Series will be $y = A + Bz^{-6} + Cz^{-12}$, &c. Alſo becauſe $8A^3 + aA^2 = 0$, it is $A = -\frac{1}{8}a$ for the firſt Coefficient.

I ſhall produce one Example more, in order to ſhew what variety of Series may be derived from the Root in ſome Equations; as alſo to ſhew all the caſes, and all the varieties that can be derived, in the preſent ſtate of the Equation. Let us therefore aſſume this Equation, $y^3 - \frac{y^2x^2}{a} + x^3 - \frac{a^3x^2}{y^2} + \frac{a^6}{y^3} - \frac{a^7}{y^2x^2} + \frac{a^6}{x^3} - \frac{a^3y^2}{x^2} + a^3 = 0$, or rather $y^3 - a^{-1}y^2x^2 + x^3 - a^3y^{-2}x^2 + a^6y^{-3} - a^7y^{-2}x^{-2} + a^6x^{-3} - a^3y^2x^{-2} + a^3 = 0$. Which if we make $y = Ax^m$, &c. and diſpoſe the Terms according to a combined Arithmetical Progreſſion, will appear thus :

$$\left.\begin{array}{ccccccc}
 & & & +a^6x^{-3} & & & \\
 & -a^3A^2x^{2m-2} & * & * & * & -a^7A^{-2}x^{-2m-2} & \\
 & * & * & * & * & * & \\
A^3x^{3m} & * & * & +a^3 & * & * & +a^6A^{-3}x^{-3m},\&c. \\
 & * & * & * & * & * & \\
 & -a^{-1}A^2x^{2m+2} & * & * & * & -a^3A^{-2}x^{-2m+2} & \\
 & & & +x^3 & & &
\end{array}\right\} = 0.$$

Now here it is plain by the diſpoſition of the Terms, that the Ruler can be apply'd eight times, and no oftner, or that there are eight caſes of external Terms to be try'd, each of which may give a Series for the Root, if the Coefficients will allow it, of which four will be aſcending, and four deſcending. And firſt for the four caſes of aſcending Series, in which the Root will converge by the aſcending Powers of x; and afterwards for the other four caſes, when the Series converges by the deſcending Powers of x.

I. Apply the Ruler, or, (which is the ſame thing,) aſſume the Equation $a^6A^{-3}x^{-3m} - a^7A^{-2}x^{-2m-2} = 0$, which will give $-3m = -2m - 2$, or $m = 2$; alſo $A = \frac{1}{a}$. The Number reſulting from theſe Indices is -6. But the Ruler in its parallel motion will next come to the Index -3; then to $-2m + 2$, or -2; then to 0; then to $2m - 2$, or 2; then to 3; and laſtly to $3m$ and $2m + 2$, or 6. But the Numbers $-6, -3, -2, 0, 2, 3, 6$, are in an aſcending Arithmetical Progreſſion, of which the common difference is 1; and therefore the form of the Series will be $y = Ax^2 + Bx^3 + Cx^4$, &c. and its firſt Term will be $\frac{x^2}{a}$.

II. Aſſume the Equation $a^6x^{-3} - a^7A^{-2}x^{-2m-2} = 0$, which will give $-3 = -2m - 2$, or $m = \frac{1}{2}$; alſo $A = \pm a^{\frac{1}{2}}$. The Number reſulting hence is -3; the next will be $-3m$, or $-1\frac{1}{2}$; the next $2m - 2$, or -1; the next 0; the next $-2m + 2$, or 1; the next $3m$, or $1\frac{1}{2}$; the two laſt $2m + 2$ and 3, are each 3. But the Numbers $-3, -1\frac{1}{2}, -1, 0, 1, 1\frac{1}{2}, 3$, will be found in an aſcending Arithmetical Progreſſion, of which the common difference is $\frac{1}{2}$; and therefore the form of the Series will be $y = Ax^{\frac{1}{2}} + Bx + Cx^{\frac{3}{2}} + Dx^2$, &c. and its firſt Term will be $\pm\sqrt{ax}$.

III. Aſſume the Equation $a^6x^{-3} - a^3A^2x^{2m-2} = 0$, which will give $-3 = 2m - 2$, or $m = -\frac{1}{2}$; alſo $A = \pm a^{\frac{3}{2}}$. The Number reſulting is -3; the next $3m$, or $-1\frac{1}{2}$; the next $-2m-2$, or -1; the next 0; the next $2m+2$, or 1; the next $-3m$, or $1\frac{1}{2}$; the two laſt 3 and $-2m+2$, which are each 3. But the Numbers -3, $-1\frac{1}{2}$, -1, 0, 1, $1\frac{1}{2}$, 3, will be all comprehended in an aſcending Arithmetical Progreſſion, of which the common difference is $\frac{1}{2}$; and therefore the form of the Series will be $y = Ax^{-\frac{1}{2}} + B + Cx^{\frac{1}{2}} + Dx$, &c. and the firſt Term will be $\pm a^{\frac{3}{2}}x^{-\frac{1}{2}}$, or $\pm a\sqrt{\frac{a}{x}}$.

IV. Aſſume the Equation $A^3x^{3m} - a^3A^2x^{2m-2} = 0$, which will give $3m = 2m - 2$, or $m = -2$; alſo $A = a^3$. The Number reſulting is -6; the next will be -3; the next $2m+2$, or -2; the next 0; the next $-2m-2$, or 2; the next 3; the two laſt $-3m$ and $-2m+2$, each of which is 6. But the Numbers -6, -3, -2, 0, 2, 3, 6, belong to an aſcending Arithmetical Progreſſion, of which the common difference is 1. Therefore the form of the Series will be $y = Ax^{-2} + Bx^{-1} + C + Dx$, &c. and its firſt Term will be $\frac{a^3}{x^2}$.

The four deſcending Series are thus derived.

I. Aſſume the Equation $A^3x^{3m} - a^{-1}A^2x^{2m+2} = 0$, which will give $3m = 2m + 2$, or $m = 2$; alſo $A = \frac{1}{a}$. The Number reſulting is 6; the next will be 3; the next $2m-2$, or 2; the next 0; the next $-2m+2$, or -2; the next -3; the two laſt $-3m$ and $-2m-2$, each of which is -6. But the Numbers 6, 3, 2, 0, -2, -3, -6, belong to a deſcending Arithmetical Progreſſion, of which the common difference is 1. Therefore the form of the Series will be $y = Ax^2 + Bx + C + Dx^{-1}$, &c. and the firſt Term will be $\frac{x^2}{a}$.

II. Aſſume the Equation $x^3 - a^{-1}A^2x^{2m+2} = 0$, which will give $2m + 2 = 3$, or $m = \frac{1}{2}$; alſo $A = \pm a^{\frac{1}{2}}$. The Number reſulting is 3; the next will be $3m$, or $1\frac{1}{2}$; the next $-2m+2$, or 1; the next 0; the next $2m-2$, or -1; the next $-3m$, or $-1\frac{1}{2}$; the two laſt -3 and $-2m-2$ are each -3. But the Numbers 3, $1\frac{1}{2}$, 1, 0, -1, $-1\frac{1}{2}$, -3, belong to a deſcending Arithmetical Progreſſion, of which the common difference is $\frac{1}{2}$. Therefore the form of the Series will be $y = Ax^{\frac{1}{2}} + Bx^0 + Cx^{-\frac{1}{2}} + Dx^{-1}$, &c. and the firſt Term will be $\pm\sqrt{ax}$.

III.

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III. Aſſume the Equation $x^3 - a^3A^{-2}x^{-2m+2} = 0$, which will give $3 = -2m+2$, or $m = -\frac{1}{2}$; alſo $A = \pm a^{\frac{3}{2}}$. The Number reſulting from hence is 3; the next will be $-3m$, or $1\frac{1}{2}$; the next $2m+2$, or 1; the next 0; the next $-2m-2$, or -1; the next $3m$, or $-1\frac{1}{2}$; the two laſt -3 and $2m-2$, each of which are -3. But the Numbers 3, $1\frac{1}{2}$, 1, 0, -1, $-1\frac{1}{2}$, -3, are comprehended in a deſcending Arithmetical Progreſſion, of which the common difference is $\frac{1}{2}$. Therefore the form of the Series will be $y = Ax^{-\frac{1}{2}} + Bx^{-1} + Cx^{-\frac{3}{2}} + Dx^{-2}$, &c. and the firſt Term will be $\pm a^{\frac{3}{2}}x^{-\frac{1}{2}}$, or $\pm a\sqrt{\frac{a}{x}}$.

IV. Laſtly, aſſume the Equation $a^6A^{-3}x^{-3m} - a^3A^{-2}x^{-2m+2} = 0$, which will give $-3m = -2m+2$, or $m = -2$; alſo $A = a^3$. The Number reſulting is 6; the next will be 3; the next $-2m-2$; or 2; the next 0; the next $2m+2$, or -2; the next -3; the two next $3m$ and $2m-2$, are each -6. But the Numbers 6, 3, 2, 0, -2, -3, -6, belong to a deſcending Arithmetical Progreſſion, of which the common difference is 1. Therefore the form of the Series will be $y = Ax^{-2} + Bx^{-3} + Cx^{-4} + Dx^{-5}$, &c. and the firſt Term is $\frac{a^3}{x^2}$.

And this may ſuffice in all Equations of this kind, for finding the forms of the ſeveral Series, and their firſt Approximations. Now we muſt proceed to their farther Reſolution, or to the Method of finding all the reſt of the Terms ſucceſſively.

SECT. V. *The Reſolution of Affected Specious Equations, proſecuted by various Methods of Analyſis.*

35. HITHERTO it has been ſhewn, when an Equation is propoſed, in order to find its Root, how the Terms of the Equation are to be diſpoſed in a two-fold regular ſucceſſion, ſo as thereby to find the initial Approximations, and the ſeveral forms of the Series in all their various circumſtances. Now the Author proceeds in like manner to diſcover the ſubſequent Terms of the Series, which may be done with much eaſe and certainty, when the form of the Series is known. For this end he finds Reſidual or Supplemental Equations, in a regular ſucceſſion alſo, the Roots of which are a continued Series of Supplements to the Root required. In every one of which Supplemental Equations the Approximation is

found, by rejecting the more remote or leſs conſiderable Terms, and ſo reducing it to a ſimple Equation, which will give a near Value of the Root. And thus the whole affair is reduced to a kind of Compariſon of the Roots of Equations, as has been hinted already. The Root of an Equation is nearly found, and its Supplement, which ſhould make it compleat, is the Root of an inferior Equation, the Supplement of which is again the Root of an inferior Equation; and ſo on for ever. Or retaining that Supplement, we may ſtop where we pleaſe.

36. The Author's Diagram, or his Proceſs of Reſolution, is very eaſy to be underſtood; yet however it may be thus farther explain'd. Having inſerted the Terms of the given Equation in the left-hand Column, (which therefore are equal to nothing, as are alſo all the ſubſequent Columns,) and having already found the firſt Approximation to the Root to be a; inſtead of the Root y he ſubſtitutes its equivalent $a+p$ in the ſeveral Terms of the Equation, and writes the Reſult over-againſt them reſpectively, in the right-hand Margin. Theſe he collects and abbreviates, writing the Reſult below, in the left-hand Column; of which rejecting all the Terms of too high a compoſition, he retains only the two loweſt Terms $4a^2p+a^2x=0$, which give $p=-\frac{1}{4}x$ for the ſecond Term of the Root. Then aſſuming $p=-\frac{1}{4}x+q$, he ſubſtitutes this in the deſcending Terms to the left-hand, and writes the Reſult in the Column to the right-hand. Theſe he collects and abbreviates, writing the Reſult below in the left-hand Column. Of which rejecting again all the higher Terms, he retains only the two loweſt $4a^2q-\frac{1}{16}ax^2=0$, which give $q=\frac{x^2}{64a}$ for the third Term of the Root. And ſo on.

Or in imitation of a former Proceſs, (which may be ſeen, pag. 165.) the Reſolution of this, and all ſuch like Equations, may be thus perform'd:

$$\begin{array}{llll} y^3+a^2y=2a^3=(\text{if } y=a+p) & a^3+3a^2p+3ap^2+p^3 & \multirow{3}{*}{$\left.\begin{array}{l}\\ \\ \\ \end{array}\right\}$} & \text{Or collecting} \\ \quad +axy+\ x^3 & +a^3+a^2p & & \text{and expunging;} \\ & +a^2x+axp & & \end{array}$$

$$\left.\begin{array}{ll} 4a^2p+3ap^2+p^3=-a^2x=(\text{if } p=-\frac{1}{4}x+q) & -a^2x\ \ +4a^2q \\ +axp \qquad\qquad +x^3 & -\frac{1}{4}ax^2\ \ +axq \\ & +\frac{3}{16}ax^2-\frac{1}{2}axq+3aq^2 \\ & -\frac{1}{64}x^3+\frac{3}{16}x^2q-\frac{3}{4}xq^2+q^3 \end{array}\right\}$$

Or collecting and expunging;

$$\begin{array}{l} 4a^2q+3aq^2+q^3=\frac{1}{16}ax^2=(\text{if } q=\frac{x^2}{64a}+r)\ \&c. \\ -\frac{1}{2}axq-\frac{3}{4}xq^2 \qquad +\frac{65}{64}x^3 \\ +\frac{3}{16}x^2q \end{array}$$

By which Proceſs the Root will be found $y=a-\frac{1}{4}x+\frac{x^2}{64a}$, &c.

Or in imitation of the Method before taught, (pag. 178, &c.) we may thus reſolve the firſt Supplemental Equation of this Example; *viz.* $4a^2p + axp + 3ap^2 + p^3 = -a^2x + x^3$; where the Terms muſt be diſpos'd in the following manner. But to avoid a great deal of unneceſſary prolixity, it may be here obſerved, that $y = a$, &c. briefly denotes, that a is the firſt Term of the Series, to be derived for the Value of y. Alſo $y = * - \frac{1}{4}x$, &c. inſinuates, that $-\frac{1}{4}x$ is the ſecond Term of the ſame Series y. Alſo $y = * * + \frac{x^2}{64a}$, &c. inſinuates, that $+\frac{x^2}{64a}$ is the third Term of the Series y, without any regard to the other Terms. And ſo for all the ſucceeding Terms; and the like is to be underſtood of all other Series whatever.

$$\left.\begin{array}{r} 4a^2p \\ \\ +axp \\ +3ap^2 \\ +p^3 \end{array}\right\} \begin{array}{l} = -a^2x \qquad * \qquad + x^3 \\ \qquad\qquad\qquad + \frac{1}{16}ax^2 + \frac{3x^3}{128} + \frac{509x^4}{4096a}, \&c. \\ \cdots\cdots\cdots - \frac{1}{4}ax^2 + \frac{1}{64}x^3 + \frac{131x^4}{512a}, \&c. \\ \cdots\cdots\cdots + \frac{3}{16}ax^2 - \frac{3x^3}{128} - \frac{1569x^4}{4096a}, \&c. \\ \cdots\cdots\cdots\cdots\cdots - \frac{1}{64}x^3 + \frac{3x^4}{1024a}, \&c. \end{array}$$

$$p = -\frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}, \&c.$$

To explain this Proceſs, it may be obſerved, that here $-a^2x$ is made the firſt Term of the Series, into which $4a^2p$ is to be reſolved; or $4a^2p = -a^2x$, &c. and therefore $p = -\frac{1}{4}x$, &c. which is ſet down below. Then is $+axp = -\frac{1}{4}ax^2$, &c. and (by ſquaring) $+3ap^2 = +\frac{3}{16}ax^2$, &c. each of which are ſet down in their proper Places. Theſe Terms being collected, will make $-\frac{1}{16}ax^2$, which with a contrary Sign muſt be ſet down for the ſecond Term of $4a^2p$; or $4a^2p = * + \frac{1}{16}ax^2$, &c. and therefore $p = * + \frac{x^2}{64a}$, &c. Then $axp = * + \frac{x^3}{64}$, &c. and (by ſquaring) $3ap^2 = * - \frac{3x^3}{128}$, &c. and (by cubing) $p^3 = -\frac{1}{64}x^3$, &c. Theſe being collected will make $-\frac{3x^3}{128}$, to be wrote down with a contrary Sign; and this, together with x^3, one of the Terms of the given Equation, will make $4a^2p = * * + \frac{131}{128}x^3$, &c. and therefore $p = * * + \frac{131x^3}{512a^2}$, &c. Then $axp = * * + \frac{131x^4}{512a}$, &c. and (by ſquaring) $3ap^2 = * *$

$-\frac{1569x^4}{4096a}$, &c. and (by cubing) $p^3 = * + \frac{3x^4}{1024a}$, &c. all which being collected with a contrary Sign, will make $4a^2p = * * * + \frac{509x^4}{4096a}$, &c. and therefore $p = * * * + \frac{509x^4}{16384a}$, &c. And by the ſame Method we may continue the Extraction as far as we pleaſe.

The *Rationale* of this Proceſs has been already deliver'd, but as it will be of frequent uſe, I ſhall here mention it again, in ſomewhat a different manner. The Terms of the Equation being duly order'd, ſo as that the Terms involving the Root, (which are to be reſolved into their reſpective Series,) being all in a Column on one ſide, and the known Terms on the other ſide; any adventitious Terms may be introduced, ſuch as will be neceſſary for forming the ſeveral Series, provided they are made mutually to deſtroy one another, that the integrity of the Equation may be thereby preſerved. Theſe adventitious Terms will be ſupply'd by a kind of Circulation, which will make the work eaſy and pleaſant enough; and the neceſſary Terms of the ſimple Powers or Roots, of ſuch Series as compoſe the Equation, muſt be derived one by one, by any of the foregoing Theorems.

Or if we are willing to avoid too many, and too high Powers in theſe Extractions, we may proceed in the following manner. The Example ſhall be the ſame Supplemental Equation as before, which may be reduced to this form, $4a^2 + ax + 3ap + pp \times p = -a^2x * + x^3$, of which the Reſolution may be thus:

$$
\begin{array}{lccccc}
 & 4a^2 & + ax & & & \\
+3ap \text{-----} & & -\frac{3}{4}ax & +\frac{3}{64}x^2 & +\frac{393x^3}{512a}, & \text{\&c.} \\
+p^2 \text{----------} & & & +\frac{1}{16}x^2 & -\frac{x^3}{128a}, & \text{\&c.} \\
\hline
 & 4a^2 & +\frac{1}{4}ax & +\frac{7}{64}x^2 & +\frac{389x^3}{512a}, & \text{\&c.} \\
p = & -\frac{1}{4}x & +\frac{x^2}{64a} & +\frac{131x^3}{512a^2} & +\frac{509x^4}{16384a^3}, & \text{\&c.} \\
\hline
 & -a^2x & * & +x^3 & * &
\end{array}
$$

The Terms $4a^2 + ax + 3ap + pp$ I call the aggregate Factor, of which I place the known part or parts $4a^2 + ax$ above, and the unknown parts $3ap + pp$ in a Column to the left-hand, ſo as that their reſpective Series, as they come to be known, may be placed regularly over-againſt them. Under theſe a Line is drawn, to receive the aggregate Series beneath it, which is form'd by the Terms of the aggregate Factor, as they become known. Under this aggregate Series comes the ſimple Factor p, or the ſymbol of the Root to be extracted, as its Terms become known alſo. Laſtly, under all are the known Terms of the Equation in their proper places. Now as theſe laſt Terms (becauſe of the Equation) are equivalent to the Product of the two Species above them; from this conſideration the Terms of the Series p are gradually derived, as follows.

Firſt, the initial Term $4a^2$ (of the aggregate Series) is brought down into its place, as having no other Term to be collected with it. Then becauſe this Term, multiply'd by the firſt Term of p, ſuppoſe q, is equal to the firſt Term of the Product, that is, $4a^2q = -a^2x$, it will be $q = -\frac{1}{4}x$, or $p = -\frac{1}{4}x$, &c. to be put down in its place. Thence we ſhall have $3ap = -\frac{3}{4}ax$, &c. which together with $+ax$ above, will make $+\frac{1}{4}ax$ for the ſecond Term of the aggregate Series. Now if we ſuppoſe r to repreſent the ſecond Term of p, and to be wrote in its place accordingly; by croſs-multiplication we ſhall have $4a^2r - \frac{1}{16}ax^2 = 0$, becauſe the ſecond Term of the Product is abſent, or $= 0$. Therefore $r = \frac{x^2}{64a}$, which may now be ſet down in its place. And hence $3ap = * + \frac{3}{64}x^2$, &c. and $p^2 = \frac{1}{16}x^2$, &c. which being collected will make $\frac{7}{64}x^2$, for the third Term of the aggregate Factor. Now if we ſuppoſe s to repreſent the third Term of p, then by croſs-multiplication, (or by our Theorem for Multiplication of infinite Series,) $4a^2s + \frac{x^3}{256} - \frac{7x^3}{256} = x^3$; (for x^3 is the third Term of the Product.) Therefore $s = \frac{131x^3}{512a^2}$, to be ſet down in its place. Then $3ap = * * + \frac{393x^3}{512a}$, &c. and $p^2 = * - \frac{x^3}{128a}$, &c. which together will make $+\frac{389x^3}{512a}$ for the fourth Term of the aggregate Series. Then putting t to repreſent the fourth Term of p, by multiplication we ſhall have $4a^2t + \frac{131x^4}{2048a} + \frac{7x^4}{4096a} - \frac{389x^4}{2048a} = 0$, whence $t = \frac{509x^4}{16384a^3}$, to be ſet down in its place. If we would proceed any farther in the Extraction, we muſt find in like manner the fourth Term of the Series $3ap$, and the third Term of p^2, in order to find the fifth Term of the aggregate Series. And thus we may eaſily and ſurely carry on the Root to what degree of accuracy we pleaſe, without any danger of computing any ſuperfluous Terms; which will be no mean advantage of theſe Methods.

Or we may proceed in the following manner, by which we ſhall avoid the trouble of raiſing any ſubſidiary Powers at all. The Supplemental Equation of the ſame Example, $4a^2p + axp + 3ap^2 + p^3 = -a^2x + x^3$, (and all others in imitation of this,) may be reduced to this form, $\overline{4a^2 + ax + \overline{3a + p} \times p} \times p = -a^2x + x^3$, which may be thus reſolved.

$$\begin{array}{rl}
 & 4a^2 + ax \\
+\overline{3a+p} & \cdots + 3a - \frac{1}{4}x + \frac{x^2}{64a}, \text{ \&c.} \\
\times p & \cdots - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2}, \text{ \&c.} \\
\hline
 & 4a^2 + \frac{1}{4}ax + \frac{7}{64}x^2 + \frac{389x^3}{512a}, \text{ \&c.} \\
\times p = & -\frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}, \text{ \&c.} \\
\hline
 & -a^2x \quad * \quad + x^3 \quad *
\end{array}$$

The Terms being diſpoſed as in this Paradigm, bring down $4a^2$ for the firſt Term of the aggregate Series, as it may ſtill be call'd, and ſuppoſe q to repreſent the firſt Term of the Series p. Then will $4a^2q = -a^2x$, or $q = -\frac{1}{4}x$, which is to be wrote every where for the firſt Term of p. Multiply $+3a$ by $-\frac{1}{4}x$ for the firſt Term of $\overline{3a+p} \times p$, with which product $-\frac{3}{4}ax$ collect the Term above, or $+ax$; the Reſult $\frac{1}{4}ax$ will be the ſecond Term of the aggregate Series. Then let r repreſent the ſecond Term of p, and we ſhall have by Multiplication $4a^2r - \frac{1}{16}ax^2 = 0$, or $r = \frac{x^2}{64a}$, to be wrote every where for the ſecond Term of p. Then as above, by croſs-multiplication we ſhall have $3a \times \frac{x^2}{64a} + \frac{1}{16}x^2 = \frac{7}{64}x^2$ for the third Term of the aggregate Series. Again, ſuppoſing s to repreſent the third Term of p, we ſhall have by Multiplication, (ſee the Theorem for that purpoſe,) $4a^2s + \frac{x^3}{256} - \frac{7x^3}{256} = x^3$, that is, $s = \frac{131x^3}{512a^2}$, to be wrote every where for the third Term of p. And by the ſame way of Multiplication the fourth Term of the aggregate Series will be found to be $\frac{389x^3}{512a}$, which will make the fourth Term of p to be $\frac{509x^4}{16384a^3}$. And ſo on.

Among all this variety of Methods for theſe Extractions, we muſt not omit to ſupply the Learner with one more, which is common and obvious enough, but which ſuppoſes the form of the Series required to be already known, and only the Coefficients to be unknown. This we may the better do here, becauſe we have already ſhewn how to determine the form and number of ſuch Series, in any caſe propoſed. This Method conſiſts in the aſſumption of a general Series for the Root, ſuch as may conveniently repreſent it, by the ſubſtitution of which in the given Equation, the general Coefficients may be determined. Thus in the preſent Equation $y^3 + axy + aay - x^3 - 2a^3 = 0$, having already found (pag. 204.) the form of the Root or Series to be $y = A + Bx + Cx^2$, &c. by the help of any of the Methods for Cubing an infinite Series, we may eaſily ſubſtitute this Series inſtead of y in this Equation, which will then become

$$\left.\begin{array}{lllll}
A^3 & + 3A^2Bx & + 3AB^2x^2 & + B^3x^3 & + 3AC^2x^4, \&c. \\
 & & + 3A^2C & + 6ABC & + 3B^2C \\
 & & & + 3A^2D & + 6ABD \\
 & & & & + 3A^2E \\
 & + aAx & + aBx^2 & + aCx^3 & + aDx^4, \&c. \\
+ a^2A & + a^2Bx & + a^2Cx^2 & + a^2Dx^3 & + a^2Ex^4, \&c. \\
- 2a^3 & * & * & - x^3 & *
\end{array}\right\} = 0.$$

Now becauſe x is an indeterminate quantity, and muſt continue ſo to be, every Term of this Equation may be ſeparately put equal to nothing, by which the general Coefficients A, B, C, D, &c. will be determined to congruous Values; and by this means the Root y will be known. Thus, (1.) $A^3 + a^2A - 2a^3 = 0$, which will give $A = a$, as before. (2.) $3A^2B + aA + a^2B = 0$, or $B = \frac{-aA}{3A^2 + a^2} = -\frac{1}{4}$. (3.) $3AB^2 + 3A^2C + aB + a^2C = 0$, or $C = -\frac{3AB^2 + aB}{3A^2 + a^2} = \frac{1}{64a}$. (4.) $B^3 + 6ABC + 3A^2D + aC + a^2D - 1 = 0$, or $D = \frac{131}{512a^2}$. (5.) $3AC^2 + 3B^2C + 6ABD + 3A^2E + aD + a^2E = 0$, or $E = \frac{509}{16384a^3}$. And ſo on, to determine F, G, H, &c. Then by ſubſtituting theſe Values of A, B, C, D, &c. in the aſſumed Root, we ſhall have the former Series $y = a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$, &c.

Or laſtly, we may conveniently enough reſolve this Equation, or any other of the ſame kind, by applying it to the general Theorem, pag. 190. for extracting the Roots of any affected Equations in Numbers. For this Equation being reduced to this form $y^3 * + \overline{a^2 + ax}$

$\times y - \overline{2a^3 + x^3} \times y^0 = 0$, we ſhall have there $m = 3$. And inſtead of the firſt, ſecond, third, fourth, fifth, &c. Coefficients of the Powers of y in the Theorem, if we write $1, 0, aa + ax, -2a^3 - x^3, 0,$ &c. reſpectively; and if we make the firſt Approximation $\frac{A}{B} = \frac{a}{1}$, or $A = a$ and $B = 1$; we ſhall have $\frac{4a^3 + x^3}{4a^2 + ax}$ for a nearer Approximation to the Root. Again, if we make $A = 4a^3 + x^3$, and $B = 4a^2 + ax$, by Subſtitution we ſhall have the Fraction $\frac{256a^9 + 96a^8x + 24a^7x^2 + 114a^6x^3 + 48a^5x^4 + 12a^4x^5 + 25a^3x^6 \cdot \cdot + 2x^9}{256a^8 + 160a^7x + 66a^6x^2 + 109a^5x^3 + 25a^4x^4 \cdot + 12a^2x^6 + 3ax^8}$ for a nearer Approximation to the Root. And taking this Numerator for A, and the Denominator for B, we ſhall approach nearer ſtill. But this laſt Approximation is ſo near, that if we only take the firſt five Terms of the Numerator, and divide them by the firſt five Terms of the Denominator, (which, if rightly managed, will be no troubleſome Operation,) we ſhall have the ſame five Terms of the Series, ſo often found already.

And the Theorem will converge ſo faſt on this, and ſuch like occaſions, that if we here take the firſt Approximation $A = a$, (making $B = 1$,) we ſhall have $y = \frac{4a^3 + x^3}{4a^2 + ax}$, &c. $= a - \frac{1}{4}x$, &c. And if again we make this the ſecond Approximation, or $A = a - \frac{1}{4}x$, (making $B = 1$,) we ſhall have $y = \frac{2A^3 + 2a^3 + x^3}{3A^2 + a^2 + ax}$, &c. $= \frac{4a^3 - \frac{3}{2}a^2x + \frac{3}{8}ax^2 + \frac{31}{32}x^3}{4a^2 - \frac{1}{2}ax + \frac{3}{16}x^2}$, &c. $= a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2}$, &c. And if again we make this the third Approximation, or $A = a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2}$, &c. (making $B = 1$,) we ſhall have the Value of the true Root to eight Terms at this Operation. For every new Operation will double the number of Terms, that were found true by the laſt Operation.

To proceed ſtill with the ſame Equation; we have found before, pag. 205, that we might likewiſe have a deſcending Series in this form, $y = Ax + B + Cx^{-1}$, &c. for the Root y, which we ſhall extract two or three ways, for the more abundant exemplification of this Doctrine. It has been already found, that $A = 1$, or that x is the firſt Approximation to the Root. Make therefore $y = x + p$, and ſubſtitute this in the given Equation $y^3 + axy + aay - x^3 - 2a^3 = 0$, which will then become $3x^2p + axp + a^2p + 3xp^2 + p^3 + ax^2 + a^2x - 2a^3 = 0$. This may be reduced to this form $\overline{3x^2 + ax + a^2 + 3xp + p^2} \times p = -ax^2 - a^2x + 2a^3$, and may be reſolved as follows.

$$
\begin{array}{rllllll}
 & 3x^2 + ax + a^2 & & & * & & \\
+ 3xp & \text{- - - - -} & -ax & -a^2 & + \frac{55a^3}{27x} & , \&c. \\
+ p^2 & \text{- - - - - - -} & & +\frac{1}{9}a^2 & + \frac{2a^3}{9x} & , \&c. \\
\hline
 & 3x^2 & * & +\frac{1}{9}a^2 & + \frac{61a^3}{27x} & , \&c. \\
p = & -\frac{1}{3}a & -\frac{a^2}{3x} & + \frac{55a^3}{81x^2} & + \frac{64a^4}{243x^3} & ; \&c. \\
\hline
 & -ax^2 & -a^2x & + 2a^3 & * & &
\end{array}
$$

The Terms of the aggregate Factor, as alſo the known Terms of the Equation, being diſpoſed as in the Paradigm, bring down $3x^2$ for the firſt Term of the aggregate Series; and ſuppoſing q to repreſent the firſt Term of the Series p, it will be $3x^2q = -ax^2$, or $q = -\frac{1}{3}a$, for the firſt Term of p. Therefore $-ax$ will be the firſt Term of $3xp$, to be put down in its place. This will make the ſecond Term of the aggregate Series to be nothing; ſo that if r repreſent the ſecond Term of p, we ſhall have by multiplication $3x^2r = -a^2x$, or $r = -\frac{a^2}{3x}$ for the ſecond Term of p, to be put down in its place. Then will $-a^2$ be the ſecond Term of $3xp$, as alſo $\frac{1}{9}a^2$ will be the firſt Term of p^2, to be ſet down each in their places. The Reſult of this Column will be $\frac{1}{9}a^2$, which is to be made the third Term of the aggregate Series. Then putting s for the third Term of p, we ſhall have by Multiplication $3x^2s - \frac{1}{27}a^3 = 2a^3$, or $s = \frac{55a^3}{81x^2}$. And thus by the next Operation we ſhall have $t = \frac{64a^4}{243x^3}$, and ſo on.

Or if we would reſolve this reſidual Equation by one of the foregoing Methods, by which the raiſing of Powers was avoided, and wherein the whole was perform'd by Multiplication alone; we may reduce it to this form, $3x^2 + ax + a^2 + 3x + p \times p \times p = -ax^2 - a^2x + 2a^3$, the Reſolution of which will be thus:

$$\begin{array}{rl}
 & 3x^2 + ax + a^2 \quad * \\
+\overline{3x+p} \text{ - - - - - -} & + 3x - \frac{1}{3}a - \frac{a^2}{3x}, \text{ \&c.} \\
\times p \text{ - - - - - -} & - \frac{1}{3}a + \frac{a^2}{3x} + \frac{55a^3}{81x^2}, \text{ \&c.} \\
\hline
\text{- - - - - - - -} & 3x^2 \quad * \quad + \frac{1}{9}a^2 + \frac{61a^3}{27x}, \text{ \&c.} \\
\times p = & - \frac{1}{3}a - \frac{a^2}{3x} + \frac{55a^3}{81x^2} + \frac{64a^4}{243x^3}, \text{ \&c.} \\
\hline
 & - ax^2 - a^2x + 2a^3 \quad *
\end{array}$$

The Terms being diſpos'd as in the Example, bring down $3x^2$ for the firſt Term of the aggregate Series, and ſuppoſing q to repreſent the firſt Term of the Series p, it will be $3x^2q = -ax^2$, or $q = -\frac{1}{3}a$. Put down $+3x$ in its proper place, and under it (as alſo after it) put down the firſt Term of p, or $-\frac{1}{3}a$, which being multiply'd, and collected with $+ax$ above, will make 0 for the ſecond Term of the aggregate Series. If the ſecond Term of p is now repreſented by r, we ſhall have $3x^2r * = -a^2x$, or $r = -\frac{a^2}{3x}$, to be put down in its ſeveral places. Then by multiplying and collecting we ſhall have $+\frac{1}{9}a^2$ for the third Term of the aggregate Series. And putting s for the third Term of p, we ſhall have by Multiplication $3x^2s - \frac{1}{27}a^3 = 2a^3$, or $s = \frac{55a^3}{81x^2}$. And ſo on as far as we pleaſe.

Laſtly, inſtead of the Supplemental Equation, we may reſolve the given Equation itſelf in the following manner:

$$\begin{array}{r|l}
y^3 & = x^3 \quad * \quad * \quad + 2a^3 \quad * \\
 & \quad - ax^2 - \frac{2}{3}a^2x + \frac{8}{3}a^3 - \frac{28a^4}{81x}, \text{ \&c.} \\
+axy & \text{- - - - -} + ax^2 - \frac{1}{3}a^2x - \frac{1}{3}a^3 + \frac{55a^4}{81x}, \text{ \&c.} \\
+a^2y & \text{- - - - - - - - - -} + a^2x - \frac{1}{3}a^3 - \frac{a^4}{3x}, \text{ \&c.} \\
\end{array}$$

$$y = x - \frac{1}{3}a - \frac{a^2}{3x} + \frac{55a^3}{81x^2} + \frac{64a^4}{243x^3}, \text{ \&c.}$$

Here becauſe it is $y^3 = x^3$, &c. it will be $y = x$, &c. and therefore $+axy = +ax^2$, &c. which muſt be ſet down in its place. Then it muſt be wrote again with a contrary ſign, that it may be $y^3 = * - ax^2$, &c. and therefore (extracting the cube-root,) $y = * - \frac{1}{3}a$, &c. Then $+a^2y = +a^2x$, &c. and $+axy = * - \frac{1}{3}a^2x$, &c.

which being collected with a contrary ſign, will make $y^3 = * * - \frac{1}{3}a^2x$, &c. and (by Extraction) $y = * * - \frac{a^2}{3x}$, &c. Hence $+ a^2y = * - \frac{1}{3}a^3$, &c. and $+ axy = * * - \frac{1}{3}a^3$, &c. which being collected with a contrary ſign, and united with $+ 2a^3$ above, will make $y^3 = * * * \frac{8}{3}a^3$, &c. whence (by Extraction) $y = * * * \frac{55a^3}{81x^2}$, &c. Then $+ a^2y = * * - \frac{a^4}{3x}$, &c. and $+ axy = * * * + \frac{55a^4}{81x}$, &c. which being collected with a contrary ſign, will make $y^3 = * * * * - \frac{28a^4}{81x}$, &c. and then (by Extraction) $y = * * * * + \frac{64a^4}{243x^3}$, &c. And ſo on.

37, 38. I think I need not trouble the Learner, or myſelf, with giving any particular Explication (or Application) of the Author's Rules, for continuing the Quote only to ſuch a certain period as ſhall be before determined, and for preventing the computation of ſuperfluous Terms; becauſe moſt of the Methods of Analyſis here deliver'd require no Rules at all, nor is there the leaſt danger of making any unneceſſary Computations.

39. When we are to find the Root y of ſuch an Equation as this, $y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5$, &c. $= z$, this is uſually call'd *the Reverſion of a Series.* For as here the Aggregate z is expreſs'd by the Powers of y; ſo when the Series is reverted, the Aggregate y will be expreſs'd by the Powers of z. This Equation, as now it ſtands, ſuppoſes z (or the Aggregate of the Series) to be unknown, and that we are to approximate to it indefinitely, by means of the known Number y and its Powers. Or otherwiſe; the unknown Number z is equivalent to an infinite Series of decreaſing Terms, expreſs'd by an Arithmetical Scale, of which the known Number y is the Root. This Root therefore muſt be ſuppoſed to be leſs than Unity, that the Series may duly converge. And thence it will follow, that z alſo will be much leſs than Unity. This is uſually called a Logarithmick Series, becauſe in certain circumſtances it expreſſes the Relation between the Logarithms and their Numbers, as will appear hereafter. If we look upon z as known, and therefore y as unknown, the Series muſt be reverted; or the Value of y muſt be expreſs'd by a Series of Terms compos'd of the known Number z and its Powers. The Author's Method for reverting this Series will be very obvious from the conſideration of his Diagram; and we ſhall meet with another Method hereafter, in another part of his Works. It will be ſufficient therefore in this place, to perform it after the manner of ſome of the foregoing Extractions.

4

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$$\left.\begin{array}{r} y \\ -\frac{1}{2}y^2 \\ +\frac{1}{3}y^3 \\ -\frac{1}{4}y^4 \\ +\frac{1}{5}y^5 \\ \&c. \end{array}\right\} \begin{array}{llllll} = z & +\frac{1}{2}z^2 & +\frac{1}{6}z^3 & +\frac{1}{24}z^4 & +\frac{1}{120}z^5, & \&c. \\ \text{- - -} & -\frac{1}{2}z^2 & -\frac{1}{2}z^3 & -\frac{7}{24}z^4 & -\frac{1}{8}z^5, & \&c. \\ \text{- - -} & \text{- - - -} & +\frac{1}{3}z^3 & +\frac{1}{2}z^4 & +\frac{5}{12}z^5, & \&c. \\ \text{- - -} & \text{- - - -} & \text{- - - -} & -\frac{1}{4}z^4 & -\frac{1}{2}z^5, & \&c. \\ \text{- - -} & \text{- - - -} & \text{- - - -} & \text{- - - -} & +\frac{1}{5}z^5, & \&c. \\ & & & & & \end{array}$$

In this Paradigm the unknown parts of the Equation are ſet down in a deſcending order to the left-hand, and the known Number z is ſet down over-againſt y to the right-hand. Then is $y = z$, &c. and therefore $-\frac{1}{2}y^2 = -\frac{1}{2}z^2$, &c. which is to be ſet down in its place, and alſo with a contrary ſign, ſo that $y = \ast + \frac{1}{2}z^2$, &c. And therefore (ſquaring) $-\frac{1}{2}y^2 = \ast - \frac{1}{2}z^3$, &c. and (cubing) $+\frac{1}{3}y^3 = +\frac{1}{3}z^3$, &c. which Terms collected with a contrary ſign, make $y = \ast\ \ast + \frac{1}{6}z^3$, &c. And therefore (ſquaring) $-\frac{1}{2}y^2 = \ast\ \ast - \frac{7}{24}z^4$, &c. and (cubing) $+\frac{1}{3}y^3 = \ast + \frac{1}{2}z^4$, &c. and $-\frac{1}{4}y^4 = -\frac{1}{4}z^4$, &c. which Terms collected with a contrary ſign, make $y = \ast\ \ast\ \ast + \frac{1}{24}z^4$, &c. Therefore $-\frac{1}{2}y^2 = \ast\ \ast\ \ast - \frac{1}{8}z^5$, &c. and $+\frac{1}{3}y^3 = \ast\ \ast + \frac{5}{12}z^5$, &c. and $-\frac{1}{4}y^4 = \ast - \frac{1}{2}z^5$, &c. and $+\frac{1}{5}y^5 = +\frac{1}{5}z^5$, &c. which Terms collected with a contrary ſign, make $y = \ast\ \ast\ \ast\ \ast + \frac{1}{120}z^5$, &c. And ſo of the reſt.

40. Thus if we were to revert the Series $y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + \frac{5}{112}y^7 + \frac{35}{1152}y^9 + \frac{63}{2816}y^{11}$, &c. $= z$, (where the Aggregate of the Series, or the unknown Number z, will repreſent the Arch of a Circle, whoſe Radius is 1, if its right Sine is repreſented by the known Number y,) or if we were to find the value of y, conſider'd as unknown, to be expreſsd by the Powers of z, now conſider'd as known; we may proceed thus:

$$\left.\begin{array}{r} y \\ +\frac{1}{6}y^3 \\ +\frac{3}{40}y^5 \\ +\frac{5}{112}y^7 \\ +\frac{35}{1152}y^9 \\ \&c. \end{array}\right\} \begin{array}{llllll} = z & -\frac{1}{6}z^3 & +\frac{1}{120}z^5 & -\frac{1}{5040}z^7 & +\frac{1}{362880}z^9, & \&c. \\ \text{- - -} & +\frac{1}{6}z^3 & -\frac{1}{12}z^5 & +\frac{13}{720}z^7 & -\frac{41}{18144}z^9, & \&c. \\ \text{- - -} & \text{- - - -} & +\frac{3}{40}z^5 & -\frac{1}{16}z^7 & +\frac{23}{960}z^9, & \&c. \\ \text{- - -} & \text{- - - -} & \text{- - - -} & +\frac{5}{112}z^7 & -\frac{35}{672}z^9, & \&c. \\ \text{- - -} & \text{- - - -} & \text{- - - -} & \text{- - - -} & +\frac{35}{1152}z^9, & \&c. \\ & & & & & \end{array}$$

The Terms being diſpoſed as you ſee here, we ſhall have $y = z$, &c. and therefore (cubing) $\frac{1}{6}y^3 = \frac{1}{6}z^3$, &c. which makes $y = \ast - \frac{1}{6}z^3$, &c. ſo that (cubing) we ſhall have $+\frac{1}{6}y^3 = \ast - \frac{1}{12}z^5$, &c. and alſo $\frac{3}{40}y^5 = \frac{3}{40}z^5$, &c. and collecting with a contrary ſign,

y

$y = * * + \frac{1}{120}z^5$, &c. Hence $\frac{1}{6}y^3 = * * \frac{13}{720}z^7$, &c. and $\frac{3}{40}y^5 = * - \frac{1}{16}z^7$, &c. and $\frac{5}{112}y^7 = \frac{5}{112}z^7$, &c. and collecting with a contrary ſign, $y = * * * - \frac{1}{5040}z^7$, &c. And ſo on.

If we ſhould deſire to perform this Extraction by another of the foregoing Methods, that is, by ſuppoſing the Equation to be reduced to this form $\overline{1 + \frac{1}{6}y^2 + \frac{3}{40}y^4 + \frac{5}{112}y^6 + \frac{35}{1152}y^8, \&c.} \times y = z$, it may be ſufficient to ſet down the Praxis, as here follows.

$$\begin{array}{llllllll}
 & 1 & & * & * & * & * & \\
+ & \frac{1}{6}y^2 & \text{----} & +\frac{1}{6}z^2 & -\frac{1}{18}z^4 & +\frac{1}{135}z^6 & -\frac{1}{1890}z^8, & \&c. \\
+ & \frac{3}{40}y^4 & \text{---------} & & +\frac{3}{40}z^4 & -\frac{1}{20}z^6 & +\frac{3}{200}z^8, & \&c. \\
+ & \frac{5}{112}y^6 & \text{---------------} & & & +\frac{5}{112}z^6 & -\frac{5}{112}z^8, & \&c. \\
+ & \frac{35}{1152}y^8 & \text{-------------------} & & & & +\frac{35}{1152}z^8, & \&c. \\
\hline
 & 1 & & +\frac{1}{6}z^2 & +\frac{7}{360}z^4 & +\frac{31}{15120}z^6 & +\frac{127}{604800}z^8, & \&c. \\
 & y = z & & -\frac{1}{6}z^3 & +\frac{1}{120}z^5 & -\frac{1}{5040}z^7 & +\frac{1}{362880}z^9, & \&c. \\
\hline
 & z & & * & * & * & * &
\end{array}$$

41. The affected Cubick Equation, which the Author here aſſumes to be ſolved, has infinite Series for the Coefficients of the Powers of y; and therefore its Terms being diſpoſed (as is taught before) according to a double Arithmetical Scale, the Roots of each of which are y and z, it will ſtand as is repreſented here below. Or taking Az^m for the firſt Approximation to the Root y, and ſubſtituting it in the firſt Table, it will appear as is here ſet down in the ſecond Table.

$$\left.\begin{array}{cccc}
* & * & * & -\;8 \\
z^2y^3 & -z^2y^2 & +z^2y & +z^2 \\
-\frac{1}{2}z^4 & +z^4 & -2z^4 & -4z^4 \\
+\frac{1}{3}z^6 & -z^6 & +3z^6 & +9z^6 \\
-\frac{1}{4}z^8 & +z^8 & -4z^8 & -16z^8 \\
\&c. & \&c. & \&c. & \&c.
\end{array}\right\} = 0. \qquad
\left.\begin{array}{cccc}
* & * & * & -\;8 \\
A^3z^{3m+2} & -A^2z^{2m+2} & +Az^{m+2} & +z^2 \\
-\frac{1}{2}A^3z^{3m+4} & +A^2z^{2m+4} & -2Az^{m+4} & -4z^4 \\
+\frac{1}{3}A^3z^{3m+6} & -A^2z^{2m+6} & +3Az^{m+6} & +9z^6 \\
\&c. & \&c. & \&c. & \&c.
\end{array}\right\} = 0.$$

Now the only caſe of external Terms, to be diſcover'd by applying the Ruler, will give the Equation $A^3z^{3m+2} - 8 = 0$, whence $3m + 2 = 0$, or $m = -\frac{2}{3}$, and the Coefficient $A = 2$. The next Number or Index, to which the Ruler in its parallel motion will apply itſelf, will be $2m + 2$, or $\frac{2}{3}$; the next will be $m + 2$, or $\frac{4}{3}$; and ſo on. Which aſcending Arithmetical Progreſſion $0, \frac{2}{3}, \frac{4}{3}$, &c. will have $\frac{2}{3}$ for its common difference. Therefore $y = Az^{-\frac{2}{3}} + B + Cz^{\frac{2}{3}} + Dz^{\frac{4}{3}} + Ez^2$, &c. will be the form of the Root in this Equation. It may be reſolved by any of the foregoing Methods, but perhaps moſt readily by ſubſtituting the Value of y now found in the given Equation, and thence determining the general Coefficients as before. By which the Root will be found to be $y = 2z^{-\frac{2}{3}} + \frac{1}{3} - \frac{1}{9}z^{\frac{2}{3}} + \frac{37}{162}z^{\frac{4}{3}} - \frac{14}{81}z^{2} + \frac{499}{1458}z^{\frac{8}{3}} + \frac{6721}{26244}z^{\frac{10}{3}}$, &c.

42. To reſolve this affected Quadratick Equation, in which one of the Coefficients is an infinite Series; if we ſuppoſe $y = Ax^{m}$, &c. we ſhall have (by Subſtitution) the Equation as it ſtands here below. Then by applying the Ruler, we ſhall have $-aAx^{m} + \frac{x^4}{4a^2} = 0$, whence $m = 4$, and $A = \frac{1}{4a^3}$. The next Index, that the Ruler in its parallel motion will arrive at, is $m+1$, or 5; the next is $m+2$, or 6; &c. ſo that the common difference of the Progreſſion is 1, and the Root may be repreſented by $y = Ax^4 + Bx^5 + Cx^6$, &c. which may be extracted as here follows.

$$\left.\begin{array}{lllll}
A^2x^{2m} & -aAx^{m} & & \&c. \\
* & -Ax^{m+1} & & * \\
* & -\frac{A}{a}x^{m+2} & & * \\
* & -\frac{A}{a^2}x^{m+3} & & * \\
* & -\frac{A}{a^3}x^{m+4} & + & \frac{x^4}{4a^2} \\
 & \&c. & &
\end{array}\right\} = 0.$$

$$\left.\begin{array}{l}
-ay \\
-xy \\
-\frac{x^2}{a}y \\
-\frac{x^3}{a^2}y \\
-\frac{x^4}{a^3}y \\
\&c. \\
+y^2
\end{array}\right\}\begin{array}{l}
= -\frac{x^4}{4a^2} + \frac{x^5}{4a^3} \quad * \quad * \quad -\frac{x^8}{16a^6},\ \&c. \\
- - - - \frac{x^5}{4a^3} + \frac{x^6}{4a^4} \quad * \quad * \\
- - - - - - -\frac{x^6}{4a^4} + \frac{x^7}{4a^5} \quad * \quad \&c. \\
- - - - - - - - - - -\frac{x^7}{4a^5} + \frac{x^8}{4a^6},\ \&c. \\
- - - - - - - - - - - - -\frac{x^8}{4a^6},\ \&c. \\
\\
- - - - - - - - - - - - - +\frac{x^8}{16a^6},\ \&c.
\end{array}$$

$$y = \frac{x^4}{4a^3} - \frac{x^5}{4a^4} \ * \ * + \frac{x^8}{16a^7},\ \&c.$$

Here becauſe it is $-ay = -\frac{x^4}{4a^2}$, &c. it will be $y = \frac{x^4}{4a^3}$, &c. Therefore $-xy = -\frac{x^5}{4a^3}$, &c. which wrote with a contrary Sign will make $-ay = * + \frac{x^5}{4a^3}$, and therefore $y = * - \frac{x^5}{4a^4}$, &c. Then $-xy = * + \frac{x^6}{4a^4}$, &c. and $-\frac{x^2}{a}y = -\frac{x^6}{4a^4}$, &c. which collected will deſtroy each other, and therefore $-ay = * * + 0$, &c. and conſequently $y = * * + 0$, &c. &c.

But there is another caſe of external Terms, which will be diſcover'd by the Ruler, and which will give $A^2x^{2m} - aAx^{m} = 0$, whence $m = 0$, and $A = a$. Here the Progreſſion of the Indices will be 0, 1, 2, &c. ſo that $y = A + Bx + Cx^2$, &c. will be the form of the Series. And if this Root be proſecuted by any of the

Methods taught before, it will be found $y = a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{3x^4}{4a^3}$, &c.

Now in the given Equation, becauſe the infinite Series $a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3}$, &c. is a Geometrical Progreſſion, and therefore is equal to $\frac{a^2}{a-x}$, as may be proved by Diviſion; if we ſubſtitute this, the Equation will become $y^2 - \frac{a^2}{a-x}y + \frac{x^4}{4a^2} = 0$. And if we extract the ſquare-root in the ordinary way, it will give $y = \frac{a^3 \pm \sqrt{a^6 - a^2x^4 + 2ax^5 - x^6}}{2a^2 - 2ax}$ for the exact Root. And if this Radical be reſolved, and then divided by this Denominator, the ſame two Series will ariſe as before, for the two Roots of this Equation. And this ſufficiently verifies the whole Proceſs.

43. In Series that are very remarkable, and of general uſe, the Law of Continuation (if not obvious) ſhould be always aſſign'd, when that can be conveniently done; which renders a Series ſtill more uſeful and elegant. This may commonly be diſcover'd in the Computation, by attending to the formation of the Coefficients, eſpecially if we put Letters to repreſent them, and thereby keep them as general as may be, deſcending to particulars by degrees. In the Logarithmic Series, for inſtance, $z = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4$, &c. the Law of Conſecution is very obvious, ſo that any Term, tho' ever ſo remote, may eaſily be aſſign'd at pleaſure. For if we put T to repreſent any Term indefinitely, whoſe order in the Series is expreſs'd by the natural Number m, then will $T = \pm\frac{1}{m}y^m$, where the Sign muſt be $+$ or $-$ according as m is an odd or an even Number. So that the hundredth Term is $-\frac{1}{100}y^{100}$, the next is $+\frac{1}{101}y^{101}$, &c. In the Reverſe of this Series, or $y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c. the Law of Continuation is thus. Let T repreſent any Term indefinitely, whoſe order in the Series is expreſs'd by m; then is $T = \frac{z^m}{1 \times 2 \times 3 \times 4 \times \text{\&c.}}$, which Series in the Denominator muſt be continued to as many Terms as there are Units in m. Or if c ſtands for the Coefficient of the Term immediately preceding, then is $T = \frac{c}{m}z^m$.

Again, in the Series $y = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$, &c. (by which the Relation between the Circular Arch and its right Sine is expreſs'd,) the Law of Continuation will be thus.

If T be any Term of the Series, whoſe order is expreſs'd by m, and if c be the Coefficient immediately before; then $T = \frac{-cz^{2m-1}}{\overline{2m-1}\times\overline{2m-2}}$. And in the Reverſe of this Series, or $z = y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + \frac{5}{112}y^7 + \frac{35}{1152}y^9$, &c. the Law of Conſecution will be thus. If T repreſents any Term, the Index of whoſe place in the Series is m, and if c be the preceding Coefficient; then $T = \frac{\overline{2m-3}\times\overline{2m-3}\times c}{\overline{2m-1}\times\overline{2m-2}}y^{2m-1}$. And the like of others.

44, 45, 46. If we would perform theſe Extractions after a more indefinite and general manner, we may proceed thus. Let the given Equation be $y^3 + a^2y + axy - 2a^3 - x^3 = 0$, the Terms of which ſhould be diſpoſed as in the Margin. Suppoſe $y = b + p$, where b is to be conceived as a near Approximation to the Root y, and p as its ſmall Supplement. When this is ſubſtituted, the Equation will ſtand as it does here. Now becauſe x and p are both ſmall quantities, the moſt conſiderable quantities are at the beginning of the Equation, from whence they proceed gradually diminiſhing, both downwards and towards the right-hand; as ought always to be ſuppos'd, when the Terms of an Equation are diſpos'd according to a double Arithmetical Scale. And becauſe inſtead of one unknown quantity y, we have here introduced two, b and p, we may determine one of them b, as the neceſſity of the Reſolution ſhall require. To remove therefore the moſt conſiderable Quantities out of the Equation, and to leave only a Supplemental Equation, whoſe Root is p; we may put $b^3 + a^2b - 2a^3 = 0$, which Equation will determine b, and which therefore henceforward we are to look upon as known. And for brevity ſake, if we put $a^2 + 3b^2 = c$, we ſhall have the Equation in the Margin.

$$\left.\begin{array}{llll} -2a^3 & +a^2y & * & +y^3 \\ * & +axy & * & * \\ * & * & * & * \\ -x^3 & * & * & * \end{array}\right\} = 0.$$

$$\left.\begin{array}{llll} \left.\begin{array}{l} -2a^3 \\ +a^2b \\ +b^3 \end{array}\right\} & \left.\begin{array}{l} +a^2p \\ +3b^2p \end{array}\right\} & +3bp^2 & +p^3 \\ +abx & +axp & * & * \\ * & * & & \\ -x^3 & * & & \end{array}\right\} = 0.$$

Now here, becauſe the two initial Terms $+cp + abx$ are the moſt conſiderable of the Equation, which might be removed, if for the firſt Approximation to p we ſhould aſſume $-\frac{abx}{c}$, and the reſulting Supplemental Equation would be depreſs'd lower; therefore make $p = -\frac{abx}{c} + q$, and by ſubſtitution we ſhall have this Equation following.

$$\left.\begin{array}{llll} * & +cp & +3bp^2 & +p^3 \\ +abx & +axp & * & \\ * & * & & \\ -x^3 & * & & \end{array}\right\} = 0.$$

Or in this Equation, if we make $\frac{3a^2b^3}{c^2} - \frac{a^2b}{c} = d$, $a - \frac{6ab^2}{c} = e$, and $\frac{a^3b^3}{c^3} + 1 = f$; it will aſſume this form.

$$\left.\begin{array}{cccc} \cdot & +cq & +3bq^2 & +q^3 \\ \cdot & \left.\begin{array}{l} -\frac{6ab^2}{c}xq \\ +axq \end{array}\right\} & -\frac{3ab}{c}xq^2 & \cdot \\ \left.\begin{array}{l} +\frac{3a^2b^3}{c^2}x^2 \\ -\frac{a^2b}{c}x^2 \end{array}\right\} & +\frac{3a^2b^2}{c^2}x^2q & \cdot & \cdot \\ \left.\begin{array}{l} -\frac{a^3b^3}{c^3}x^3 \\ -x^3 \end{array}\right\} & \cdot & \cdot & \cdot \end{array}\right\} = 0.$$

Here becauſe the Terms to be next removed are $+cq + dx^2$, we may put $q = -\frac{d}{c}x^2 + r$, and by Subſtitution we ſhall have another Supplemental Equation, which will be farther depreſs'd, and ſo on as far as we pleaſe. Therefore

$$\left.\begin{array}{cccc} \cdot & +cq & +3bq^2 & +q^3 \\ \cdot & +exq & -\frac{3ab}{c}xq^2 & \cdot \\ +dx^2 & +\frac{3a^2b^2}{c^2}x^2q & \cdot & \cdot \\ -fx^3 & \cdot & & \end{array}\right\} = 0.$$

we ſhall have the Root $y = b - \frac{ab}{c}x - \frac{d}{c}x^2$, &c. where b will be the Root of this Equation $b^3 + a^2b - 2a^3 = 0$, $c = a^2 + 3b^2$, $d = \frac{3a^2b^3}{c^2} - \frac{a^2b}{c}$, $e = a - \frac{6ab^2}{c}$, $f = \frac{a^3b^3}{c^3} + 1$, &c.

Or by another Method of Solution, if in this Equation we aſſume (as before) $y = A + Bx + Cx^2 + Dx^3$, &c. and ſubſtitute this in the Equation, to determine the general Coefficients, we ſhall have $y = A - \frac{aA}{c^2}x + \frac{a^4A}{c^6}x^2 + \frac{c^8 + 7a^5A^3 - a^7A}{c^{10}}x^3$, &c. wherein A is the Root of the Equation $A^3 + a^2A - 2a^3 = 0$, and $c^2 = 3A^2 + a^2$.

47. All Equations cannot be thus immediately reſolved, or their Roots cannot always be exhibited by an Arithmetical Scale, whoſe Root is one of the Quantities in the given Equation. But to perform the Analyſis it is ſometimes required, that a new Symbol or Quantity ſhould be introduced into the Equation, by the Powers of which the Root to be extracted may be expreſs'd in a converging Series. And the Relation between this new Symbol, and the Quantities of the Equation, muſt be exhibited by another Equation. Thus if it were propoſed to extract the Root y of this Equation, $x = a + y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4$, &c. it would be in vain to expect, that it might be expreſs'd by the ſimple Powers of either x or a. For the Series itſelf ſuppoſes, in order to its converging, that y is ſome ſmall Number leſs than Unity; but x and a are under no ſuch limitations. And therefore a Series, compoſed of the aſcending Powers of x, may be a diverging Series. It is therefore neceſſary to introduce a new Symbol, which ſhall alſo be ſmall, that a Series [illegible]

form'd of its Powers may converge to y. Now it is plain, that x and a, tho' ever ſo great, muſt always be near each other, becauſe their difference $y - \frac{1}{2}y^2$, &c. is a ſmall quantity. Aſſume therefore the Equation $x - a = z$, and z will be a ſmall quantity as required; and being introduced inſtead of $x - a$, will give $z = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4$, &c. whoſe Root being extracted will be $y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4$, &c. as before.

48. Thus if we had the Equation $y^3 + y^2 + y - x^3 = 0$, to find the Root y; we might have a Series for y compoſed of the aſcending Powers of x, which would converge if x were a ſmall quantity, leſs than Unity, but would diverge in contrary Circumſtances. Suppoſing then that x was known to be a large Quantity; in this caſe the Author's Expedient is this. Making z the Reciprocal of x, or ſuppoſing the Equation $x = \frac{1}{z}$, inſtead of x he introduces z into the Equation, by which means he obtains a converging Series, conſiſting of the Powers of z aſcending in the Numerators, that is in reality, of the Powers of x aſcending in the Denominators. This he does, to keep within the Caſe he propoſed to himſelf; but in the Method here purſued, there is no occaſion to have recourſe to this Expedient, it being an indifferent matter, whether the Powers of the converging quantity aſcend in the Numerators or the Denominators.

Thus in the given Equation $\left.\begin{matrix} y^3 + y^2 + y & * \\ & -x^3 \end{matrix}\right\} = 0$, or (making $y = Ax^m$, &c.) $\left.\begin{matrix} A^3x^{3m} + A^2x^{2m} + Ax^m & *, \text{ \&c.} \\ & -x^3 \end{matrix}\right\} = 0$, by applying the Ruler we ſhall have the exterior Terms $A^3x^{3m} - x^3 = 0$, or $m = 1$, and $A = 1$. Alſo the reſulting Number or Index is 3. The next Term to which the Ruler approaches will give $2m$, or 2; the laſt m, or 1. But 3, 2, 1, make a deſcending Progreſſion, of which the common difference is 1. Therefore the form of the Root will be $y = Ax + B + Cx^{-1} + Dx^{-2}$, &c. which we may thus extract.

$$\left.\begin{array}{r} y^3 \\ +y^2 \\ +y \end{array}\right\} \begin{array}{l} = x^3 - x^2 - \frac{1}{3}x + \frac{2}{3}x^0 + \frac{7}{81}x^{-1}, \text{ \&c.} \\ \text{- - - - } + x^2 - \frac{2}{3}x - \frac{1}{3}x^0 + \frac{11}{81}x^{-1}, \text{ \&c.} \\ \text{- - - - - - - - } + x - \frac{1}{3}x^0 - \frac{2}{9}x^{-1} + \frac{7}{81}x^{-2} + \frac{5}{81}x^{-3}, \text{ \&c.} \end{array}$$

Becauſe $y^3 = x^3$, &c. it will be $y = x$, &c. and therefore $y^2 = x^2$, &c. which will make $y^3 = * - x^2$, &c. and (by Extraction) $y = * - \frac{1}{3}$, &c. Then (by ſquaring) $y^2 = * - \frac{2}{3}x$, &c. which with x below, and changing the Sign, makes $y^3 = * * - \frac{1}{3}x$, &c. and therefore

y

$y = * * - \frac{2}{9}x^{-1}$, &c. Then $y^2 = * * - \frac{1}{3}$, &c. and $y = * - \frac{1}{3}$, &c. which together, changing the Sign, make $y^3 = * * * + \frac{2}{3}$, &c. and $y = * * * + \frac{7}{81}x^{-2}$, &c. Then $y^2 = * * * + \frac{11}{81}x^{-1}$, &c. and $y = * * - \frac{2}{9}x^{-1}$, &c. and therefore $y^3 = * * * * + \frac{7}{81}x^{-1}$, &c. and $y = * * * * + \frac{5}{81}x^{-3}$, &c.

Now as this Series is accommodated to the caſe of convergency when x is a large Quantity, ſo we may derive another Series from hence, which will be accommodated to the caſe when x is a ſmall quantity. For the Ruler will direct us to the external Terms $Ax^m - x^3 = 0$, whence $m = 3$, and $A = 1$; and the reſulting Number is 3. The next Term will give $2m$, or 6; and the laſt is $3m$, or 9. But 3, 6, 9 will form an aſcending Progreſſion, of which the common difference is 3. Therefore $y = Ax^3 + Bx^6 + Cx^9$, &c. will be the form of the Series in this caſe, which may be thus derived.

$$\left.\begin{array}{r} y \\ +y^2 \\ +y^3 \end{array}\right\} \begin{array}{lllllll} = x^3 & - x^6 & + x^9 & * & - 4x^{15} & + 14x^{18}, & \text{\&c.} \\ \text{- - - - -} & + x^6 & - 2x^9 & + 3x^{12} & - 2x^{15} & - 7x^{18}, & \text{\&c.} \\ \text{- - - - -} & \text{- - - - -} & + x^9 & - 3x^{12} & + 6x^{15} & - 7x^{18}, & \text{\&c.} \end{array}$$

Here becauſe $y = x^3$, &c. it will be $y^2 = x^6$, &c. and therefore $y = * - x^6$, &c. Then $y^2 = * - 2x^9$, &c. and $y^3 = x^9$, &c. and therefore $y = * * + x^9$, &c. Then $y^2 = * * + 3x^{12}$, &c. and $y^3 = * - 3x^{12}$, &c. and therefore $y = * * * + 0$, &c.

The Expedient of the Ruler will indicate a third caſe of external Terms, which may be try'd alſo. For we may put $A^3x^{3m} + A^2x^{2m} + Ax^m = 0$, whence $m = 0$, and the Number reſulting from the other Term is 3. Therefore 3 will be the common difference of the Progreſſion, and the form of the Root will be $y = A + Bx^3 + Cx^6$, &c. But the Equation $A^3 + A^2 + A = 0$, will give $A = 0$, which will reduce this to the former Series. And the other two Roots of the Equation will be impoſſible.

If the Equation of this Example $y^3 + y^2 + y - x^3 = 0$ be multiply'd by the factor $y - 1$, we ſhall have the Equation $y^4 - y - x^3y + x^3 = 0$, or $\left.\begin{array}{l} y^4 * * - y \quad\quad * \\ \quad\quad\quad - x^3y + x^3 \end{array}\right\} = 0$, which when reſolved, will only afford the ſame Series for the Root y as before.

49. This Equation $y^4 - x^2y^2 + xy^2 + 2y^2 - 2y + 1 = 0$, when reduced to the form of a double Arithmetical Scale, will ſtand as in the Margin.

$$\left.\begin{matrix} y^4 \ * + 2y^2 - 2y + 1 \\ + xy^2 \\ - x^2y^2 \end{matrix}\right\} = 0.$$

Or making $y = Ax^m$, &c.

$$\left.\begin{matrix} A^4x^{4m} \ * + 2A^2x^{2m} - 2Ax^m + 1 \\ + A^2x^{2m+1} \quad \&c. \\ - A^2x^{2m+2} \end{matrix}\right\} = 0.$$

Now the firſt Caſe of external Terms, ſhewn by the Ruler, in order for an aſcending Series, will make $A^4x^{4m} + 2A^2x^{2m} - 2Ax^m + 1 = 0$, or $m = 0$; where the reſulting Number is alſo 0. The ſecond is $2m + 1$, or 1; the third $2m + 2$, or 2. Therefore the Arithmetical Progreſſion will be 0, 1, 2, whoſe common difference is 1; and conſequently it will be $y = A + Bx + Cx^2 + Dx^3$, &c. But the Equation $A^4 + 2A^2 - 2A + 1 = 0$, which ſhould give the Value of the firſt Coefficient, will ſupply us with none but impoſſible Roots; ſo that y, the Root of this Equation, cannot be expreſs'd by an Arithmetical Scale whoſe Root is x, or by an aſcending Series that converges by the Powers of x, when x is a ſmall quantity.

As for deſcending Series, there are two caſes to be try'd; firſt the Ruler will give us $A^4x^{4m} - A^2x^{2m+2} = 0$, whence $4m = 2m + 2$, or $m = 1$, and $A = \pm 1$. The Number ariſing is 4; the next will be $2m + 1$, or 3; the next $2m$, or 2; the next m, or 1; the laſt 0. But the Arithmetical Progreſſion 4, 3, 2, 1, 0, has 1 for its common difference, and therefore the form of the Series will be $y = Ax + B + Cx^{-1}$, &c. But to extract this Series by our uſual Method, it will be beſt to reduce the Equation to this form, $y^2 - x^2 + x + 2 - 2y^{-1} + y^{-2} = 0$, and then to proceed thus:

$$\left.\begin{matrix} y^2 \\ -2y^{-1} \\ +y^{-2} \end{matrix}\right\} \begin{matrix} = x^2 - x - 2 + 2x^{-1} - \frac{3}{2}x^{-2}, \ \&c. \\ \text{- - - - - - - - - - -} - 2x^{-1} + \frac{1}{2}x^{-2}, \ \&c. \\ \text{- - - - - - - - - - - - - - - -} + x^{-2}, \ \&c. \end{matrix}$$

$$y = x - \frac{1}{2} - \frac{9}{8x} + \frac{7}{16x^2} - \frac{177}{128x^3}, \ \&c.$$

Becauſe $y^2 = x^2 - x - 2$, &c. 'tis therefore (by Extraction) $y = x - \frac{1}{2} - \frac{9}{8}x^{-1}$, &c. Then (by Diviſion) $-2y^{-1} = -2x^{-1}$, &c. ſo that $y^2 = * \ * \ * + 2x^{-1}$, &c. and (by Extraction) $y = * \ * \ * + \frac{7}{16}x^{-2}$, &c. Then $-2y^{-1} = * + \frac{1}{2}x^{-2}$, &c. and $y^{-2} = x^{-2}$, &c. which being united with a contrary ſign, make $y^2 = * \ * \ * \ * - \frac{3}{2}x^{-2}$, &c. and therefore by Extraction $y = * \ * \ * \ * - \frac{177}{128}x^{-3}$, &c.

In the other caſe of a deſcending Series we ſhall have the Equation $-A^2x^{2m+2} + 1 = 0$, whence $2m + 2 = 0$, or $m = -1$, and $A = \pm 1$. The Number hence ariſing is 0; the next will be $2m + 1$, or -1; the next $2m$, or -2; and the laſt $4m$, or -4. But the Numbers $0, -1, -2, -4$, will be found in a deſcending Arithmetical Progreſſion, the common difference of which is 1. Therefore the form of the Root is $y = Ax^{-1} + Bx^{-2} + Cx^{-3}$, &c. and the Terms of the Equation muſt be thus diſpoſed for Reſolution.

$$\begin{array}{r} y^{-2} \\ \\ -2y^{-1} \\ +\ y^{2} \end{array}\left\}\begin{array}{l} = x^{2} - x - 2 - \frac{5}{4}x^{-1} - \frac{13}{8}x^{-2}, \text{ \&c.} \\ \qquad + 2x + 1 \\ - - - - 2x - 1 + \frac{5}{4}x^{-1} + \frac{5}{8}x^{-2}, \text{ \&c.} \\ - - - - - - - - - - - - - - - + x^{-2}, \text{ \&c.} \end{array}\right.$$

$$y = \frac{1}{x} - \frac{1}{2x^{2}} + \frac{7}{8x^{3}} - \frac{7}{16x^{4}} + \frac{187}{128x^{5}}, \text{ \&c.}$$

Here becauſe it is $y^{-2} = x^{2}$, &c. it will be by Extraction of the Square-root $y^{-1} = x$, &c. and by finding the Reciprocal, $y = x^{-1}$, &c. Then becauſe $-2y^{-1} = -2x$, &c. this with a contrary Sign, and collected with $-x$ above, will make $y^{-2} = * + x$, &c. which (by Extraction) makes $y^{-1} = * + \frac{1}{2}$, &c. and by taking the Reciprocal, $y = * - \frac{1}{2}x^{-2}$, &c. Then becauſe $-2y^{-1} = * - 1$, &c. this with a contrary ſign, and collected with -2 above, will make $y^{-2} = * * - 1$, &c. and therefore (by Extraction) $y^{-1} = * * - \frac{5}{8}x^{-1}$, &c. and (by Diviſion) $y = * * + \frac{7}{8}x^{-3}$, &c. Then becauſe $-2y^{-1} = * * + \frac{5}{4}x^{-1}$, it will be $y^{-2} = * * * - \frac{5}{4}x^{-1}$, &c. and $y^{-1} = * * * - \frac{5}{8}x^{-1}$, &c. and $y = * * * - \frac{7}{16}x^{-3}$, &c. Then becauſe $-2y^{-1} = * * * + \frac{5}{8}x^{-2}$, &c. and $y^{2} = x^{-2}$, &c. theſe collected with a contrary ſign will make $y^{-2} = * * * * - \frac{13}{8}x^{-2}$, &c. and $y^{-1} = * * * * - \frac{5}{16}x^{-2}$, &c. and $y = * * * * + \frac{187}{128}x^{-4}$, &c.

Theſe are the two deſcending Series, which may be derived for the Root of this Equation, and which will converge by the Powers of x, when it is a large quantity. But if x ſhould happen to be ſmall, then in order to obtain a converging Series, we much change the Root of the Scale. As if it were known that x differs but little from 2, we may conveniently put z for that ſmall difference, or we may aſſume the Equation $x - 2 = z$. That is, inſtead of x in this Equation ſubſtitute $z + 2$, and we ſhall have a new Equation $y^{4} - z^{2}y^{2} - 3zy^{2} - 2y + 1 = 0$, which will appear as in the Margin.

Here

$$\left.\begin{array}{llll} y^4 & * & * & -2y+1 \\ & -3zy^2 & & \\ & -z^2y^2 & & \end{array}\right\} = 0.$$

Or making $y = Az^m$, *&c.*

$$\left.\begin{array}{lllll} A^4z^{4m} & * & * & -2Az^m+1 & \\ & -3A^2z^{2m+1} & & & \\ & -A^2z^{2m+2} & & & \&c. \end{array}\right\} = 0.$$

Here to have an aſcending Series, we muſt put $A^4z^{4m} - 2Az^m + 1 = 0$, whence $m = 0$, and $A = 1$. The Number hence ariſing is 0; the next is $2m + 1$, or 1; and the laſt $2m + 2$, or 2. But 0, 1, 2, are in an aſcending Progreſſion, whoſe common difference is 1. Therefore the form of the Series is $y = A + Bz + Cz^2 + Dz^3$, &c. And if the Root y be extracted by any of the foregoing Methods, it will be found $y = 1 + \frac{3}{2}z - \frac{7}{4}z^2$, &c. Alſo we may hence find two deſcending Series, which would converge by the Root of the Scale z, if it were a large quantity.

50, 51. Our Author has here opened a large field for the Solution of theſe Equations, by ſhewing, that the indeterminate quantity, or what we call the Root of the Scale, or the converging quantity, may be changed a great variety of ways, and thence new Series will be derived for the Root of the Equation, which in different circumſtances will converge differently, ſo that the moſt commodious for the preſent occaſion may always be choſe. And when one Series does not ſufficiently converge, we may be able to change it for another that ſhall converge faſter. But that we may not be left to uncertain Interpretations of the indeterminate quantity, or be obliged to make Suppoſitions at random; he gives us this Rule for finding initial Approximations, that may come at once pretty near the Root required, and therefore the Series will converge apace to it. Which Rule amounts to this: We are to find what quantities, when ſubſtituted for the indefinite Species in the propoſed Equation, will make it diviſible by the radical Species, increaſed or diminiſhed by another quantity, or by the radical Species alone. The ſmall difference that will be found between any one of thoſe quantities, and the indeterminate quantity of the Equation, may be introduced inſtead of that indeterminate quantity, as a convenient Root of the Scale, by which the Series is to converge.

Thus if the Equation propoſed be $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, and if for x we here ſubſtitute a, we ſhall have the Terms $y^3 + 2a^2y - 3a^3$, which are diviſible by $y - a$, the Quotient being $y^2 + ay + 3a^2$. Therefore we may ſuppoſe, by the foregoing Rule, that $a - x = z$ is but a ſmall quantity, or inſtead of x we may ſubſtitute $a - z$ in the propoſed Equation, which will then become $y^3 + 2a^2y - azy + 3a^2z - 3az^2 + z^3 - 2a^3 = 0$. A

Series derived from hence, compoſed of the aſcending Powers of z, muſt converge faſt, *cæteris paribus*, becauſe the Root of the Scale z is a ſmall quantity.

Or in the ſame Equation, if for x we ſubſtitute $-a$, we ſhall have the Terms $y^3 - a^3$, which are diviſible by $y - a$, the Quotient being $y^2 + ay + a^2$. Therefore we may ſuppoſe the difference between $-a$ and x to be but little, or that $-a - x = z$ is a ſmall quantity, and therefore inſtead of x we may ſubſtitute its equal $-a - z$ in the given Equation. This will then become $y^3 - azy + 3a^2z + 3az^2 - a^3 = 0$, where the Root y will converge by the Powers of the ſmall quantity z.

Or if for x we ſubſtitute $-2a$, we ſhall have the Terms $y^3 - a^2y + 6a^3$, which are diviſible by $y + 2a$, the Quotient being $y^2 - 2ay + 3a^2$. Wherefore we may ſuppoſe there is but a ſmall difference between $-2a$ and x, or that $-2a - x = z$ is a ſmall quantity; and therefore inſtead of x we may introduce its equal $-2a - z$ into the Equation, which will then become $y^3 - a^2y - azy + 6a^3 + 12a^2z + 6az^2 + z^3 = 0$.

Laſtly, if for x we ſubſtitute $-2^{\frac{1}{3}}a$, we ſhall have the Terms $y^3 - 2^{\frac{1}{3}}a^2y + a^2y$, which are diviſible by y, the Radical Species alone. Wherefore we may ſuppoſe there is but a ſmall difference between $-2^{\frac{1}{3}}a$ and x, or that $-2^{\frac{1}{3}}a - x = z$ is a ſmall quantity; and therefore inſtead of x we may ſubſtitute its equal $-2^{\frac{1}{3}}a - z$, which will reduce the Equation to $y^3 + \overline{1 - \sqrt[3]{2}} \times a^2y - azy + 3\sqrt[3]{4} \times a^2z + 3\sqrt[3]{2} \times az^2 + z^3 = 0$, wherein the Series for the Root y may converge by the Powers of the ſmall quantity z.

But the reaſon of this Operation ſtill remains to be inquired into, which I ſhall endeavour to explain from the preſent Example. In the Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, the indeterminate quantity x, of its own nature, muſt be ſuſceptible of all poſſible Values; at leaſt, if it had any limitations, they would be ſhew'd by impoſſible Roots. Among other values, it will receive theſe, a, $-a$, $-2a$, $-2^{\frac{1}{3}}a$, &c. in which caſes the Equation would become $y^3 + 2a^2y - 3a^3 = 0$, $y^3 - a^3 = 0$, $y^3 - a^2y + 6a^3 = 0$, $y^3 - 2^{\frac{1}{3}}a^2y + a^2y = 0$, &c. reſpectively. Now as theſe Equations admit of juſt Roots, as appears by their being diviſible by $y +$ or $-$ another quantity, and the laſt by y alone; ſo that in the Reſolution, the whole Equation (in thoſe caſes) would be immediately exhauſted: And in other caſes, when x does not much recede from one of thoſe

Values, the Equation would be nearly exhauſted. Therefore the introducing of z, which is the ſmall difference between x and any one of thoſe Values, muſt depreſs the Equation; and z itſelf muſt be a convenient quantity to be made the Root of the Scale, or the converging Quantity.

I ſhall give the Solution of one of the Equations of theſe Examples, which ſhall be this, $y^3 - azy + 3a^2z + 3az^2 - a^3 = 0$, or

$$\left.\begin{array}{llll} y^3 & * & * & -a^3 \\ & -azy & +3a^2z & \\ & & +3az^2 & \end{array}\right\} = 0. \qquad \begin{array}{r} y^3 \\ \\ -azy \end{array}\left\}\begin{array}{l} = a^3 - 3a^2z - 3az^2. \\ \quad + a^2z - \frac{2}{3}az^2 - \frac{5}{3}z^3, \text{ \&c.} \\ - - - - a^2z + \frac{2}{3}az^2 + \frac{5}{3}z^3, \text{ \&c.} \end{array}\right.$$

$$y = a - \tfrac{2}{3}z - \frac{5z^2}{3a} - \frac{217z^3}{81a^2}, \text{ \&c.}$$

Here becauſe $y^3 = a^3$, &c. it will be $y = a$, &c. Then $-azy = -a^2z$, &c. which muſt be wrote again with a contrary ſign, and united with $-3a^2z$ above, to make $y^3 = * - 2a^2z$, &c. and therefore $y = * - \frac{2}{3}z$, &c. Then $-azy = * + \frac{2}{3}az^2$, &c. and $y^3 = * * - \frac{11}{3}az^2$, &c. and $y = * * - \frac{5z^2}{3a}$, &c. Then $-azy = * * + \frac{5}{3}z^3$, &c. and $y^3 = * * * - \frac{5}{3}z^3$, &c. and $y = * * * - \frac{217z^3}{81a^2}$, &c.

The Author hints at many other ways of deriving a variety of Series from the ſame Equation; as when we ſuppoſe the afore-mention'd difference z to be indefinitely great, and from that Suppoſition we find Series, in which the Powers of z ſhall aſcend in the Denominators. This Caſe we have all along purſued indiſcriminately with the other Caſe, in which the Powers of the converging quantity aſcend in the Numerators, and therefore we need add nothing here about it. Another Expedient is, to aſſume for the converging quantity ſome other quantity of the Equation, which then may be conſider'd as indeterminate. So here, for inſtance, we may change a into x, and x into a. Or laſtly, to aſſume any Relation at pleaſure, (ſuppoſe $x = az + bz^2$, $x = \frac{a}{b+z}$, $x = \frac{a+cz}{b+z}$, &c.) between the indeterminate quantity of the Equation x, and the quantity z we would introduce into its room, by which new equivalent Equations may be form'd, and then their Roots may be extracted. And afterwards the value of z may be expreſs'd by x, by means of the aſſumed Equation.

52. The Author here, in a ſummary way, gives us a *Rationale* of his whole Method of Extractions, proving *à priori*, that the Series thus form'd, and continued *in infinitum*, will then be the juſt Roots of the propoſed Equation. And if they are only continued to a competent number of Terms, (the more the better,) yet then will they be a very near Approximation to the juſt and compleat Roots. For, when an Equation is propoſed to be reſolved, as near an Approach is made to the Root, ſuppoſe y, as can be had in a ſingle Term, compoſed of the quantities given by the Equation; and becauſe there is a Remainder, a Reſidual or Secondary Equation is thence form'd, whoſe Root p is the Supplement to the Root of the given Equation, whatever that may be. Then as near an approach is made to p, as can be done by a ſingle Term, and a new Reſidual Equation is form'd from the Remainder, wherein the Root q is the Supplement to p. And by proceeding thus, the Reſidual Equations are continually depreſs'd, and the Supplements grow perpetually leſs and leſs, till the Terms at laſt are leſs than any aſſignable quantities. We may illuſtrate this by a familiar Example, taken from the uſual Method of Diviſion of Decimal Fractions. At every Operation we put as large a Figure in the Quotient, as the Dividend and Diviſor will permit, ſo as to leave the leaſt Remainder poſſible. Then this Remainder ſupplies the place of a new Dividend, which we are to exhauſt as far as can be done by one Figure, and therefore we put the greateſt number we can for the next Figure of the Quotient, and thereby leave the leaſt Remainder we can. And ſo we go on, either till the whole Dividend is exhauſted, if that can be done, or till we have obtain'd a ſufficient Approximation in decimal places or figures. And the ſame way of Argumentation, that proves our Author's Method of Extraction, may eaſily be apply'd to the other ways of Analyſis that are here found.

53, 54. Here it is ſeaſonably obſerved, that tho' the indefinite Quantity ſhould not be taken ſo ſmall, as to make the Series converge very faſt, yet it would however converge to the true Root, tho' by more ſteps and ſlower degrees. And this would obtain in proportion, even if it were taken never ſo large, provided we do not exceed the due Limits of the Roots, which may be diſcover'd, either from the given Equation, or from the Root when exhibited by a Series, or may be farther deduced and illuſtrated by ſome Geometrical Figure, to which the Equation is accommodated.

So if the given Equation were $yy = ax - xx$, it is eaſy to obſerve, that neither y nor x can be infinite, but they are both liable to ſeveral Limitations. For if x be ſuppos'd infinite, the Term ax would vaniſh in reſpect of $-xx$, which would give the Value of yy impoſſible on this Suppoſition. Nor can x be negative; for then the Value of yy would be negative, and therefore the Value of y would again become impoſſible. If $x=0$, then is $y=0$ alſo; which is one Limitation of both quantities. As yy is the difference between ax and xx, when that difference is greateſt, then will yy, and conſequently y, be greateſt alſo. But this happens when $x=\frac{1}{2}a$, as alſo $y=\frac{1}{2}a$, as may appear from the following Prob. 3. And in general, when y is expreſs'd by any number of Terms, whether finite or infinite, it will then come to its Limit when the difference is greateſt between the affirmative and negative Terms; as may appear from the ſame Problem. This laſt will be a Limitation for y, but not for x. Laſtly, when $x=a$, then $y=0$; which will limit both x and y. For if we ſuppoſe x to be greater than a, the negative Term will prevail over the affirmative, and give the Value of yy negative, which will make the Value of y impoſſible. So that upon the whole, the Limitations of x in this Equation will be theſe, that it cannot be leſs than 0, nor greater than a, but may be of any intermediate magnitude between thoſe Limits.

Now if we reſolve this Equation, and find the Value of y in an infinite Series, we may ſtill diſcover the ſame Limitations from thence. For from the Equation $yy=ax-xx$, by extracting the ſquare-root, as before, we ſhall have $y=a^{\frac{1}{2}}x^{\frac{1}{2}}-\frac{x^{\frac{3}{2}}}{2a^{\frac{1}{2}}}-\frac{x^{\frac{5}{2}}}{8a^{\frac{3}{2}}}-\frac{x^{\frac{7}{2}}}{16a^{\frac{5}{2}}}$, c. that is, $y=a^{\frac{1}{2}}x^{\frac{1}{2}}$ into $1-\frac{x}{2a}-\frac{x^2}{8a^2}-\frac{x^3}{16a^3}$, &c. Here x cannot be negative; for then $x^{\frac{1}{2}}$ would be an impoſſible quantity. Nor can x be greater than a; for then the converging quantity $\frac{x}{a}$, or the Root of the Scale by which the Series is expreſs'd, would be greater than Unity, and conſequently the Series would diverge, and not converge as it ought to do. The Limit between converging and diverging will be found, by putting $x=a$, and therefore $y=0$; in which caſe we ſhall have the identical Numeral Series $1=\frac{1}{2}+\frac{1}{8}+\frac{1}{16}$, &c. of the ſame nature with ſome of thoſe, which we have elſewhere taken notice of. So that we may take x of any intermediate Value between 0 and a, in order to have a converging Series. But the nearer it is taken to the Limit 0, ſo much faſter the Series will converge to the true Root; and the nearer it is taken to the Limit a, it will converge ſo much the ſlower. But it will however

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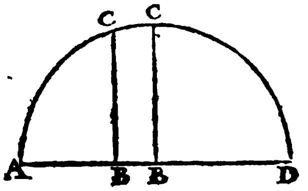
however converge, if x be taken never ſo little leſs than a. And by Analogy, a like Judgment is to be made in all other caſes.

The Limits and other affections of y are likewiſe diſcoverable from this Series. When $x = 0$, then $y = 0$. When x is a naſcent quantity, or but juſt beginning to be poſitive, all the Terms but the firſt may be neglected, and y will be a mean proportional between a and x. Alſo $y = 0$, when the affirmative Term is equal to all the negative Terms, or when $1 = \frac{x}{2a} + \frac{x^2}{8a^2} + \frac{x^3}{16a^3}$, &c. that is, when $x = a$. For then $1 = \frac{1}{2} + \frac{1}{8} + \frac{1}{16}$, &c. as above. Laſtly, y will be a *Maximum* when the difference between the affirmative Term and all the negative Terms is greateſt, which by Prob. 3. will be found when $x = \frac{1}{2}a$.

Now the Figure or Curve that may be adapted to this Equation, and to this Series, and which will have the ſame Limitations that they have, is the Circle ACD, whoſe Diameter is AD $= a$, its Abſciſs AB $= x$, and its perpendicular Ordinate BC $= y$. For as the Ordinate BC $= y$ is a mean proportional between the Segments of the Diameter AB $= x$ and BD $= a - x$, it will be $yy = ax - xx$. And therefore the Ordinate BC $= y$ will be expreſs'd by the foregoing Series. But it is plain from the nature of the Circle, that the Abſciſs AB cannot be extended backwards, ſo as to become negative; neither can it be continued forwards beyond the end of the Diameter D. And that at A and D, where the Diameter begins and ends, the Ordinate is nothing. And the greateſt Ordinate is at the Center, or when AB $= \frac{1}{2}$AD.



SECT. VI. *Tranſition to the Method of Fluxions.*

55. THE learned and ſagacious Author having thus accompliſh'd one part of his deſign, which was, to teach the Method of converting all kinds of Algebraic Quantities into ſimple Terms, by reducing them to infinite Series: He now goes on to ſhew the uſe and application of this Reduction, or of theſe Series, in the Method of Fluxions, which is indeed the principal deſign of this Treatiſe. For this Method has ſo near a connexion with, and dependence upon the foregoing, that it would be very lame and defective without it. He lays down the fundamental Principles of

this Method in a very general and ſcientifick manner, deducing them from the received and known laws of local Motion. Nor is this inverting the natural order of Science, as ſome have pretended, by introducing the Doctrine of Motion into pure Geometrical Speculations. For Geometrical and Analytical Quantities are beſt conceived as generated by local Motion; and their properties may as well be derived from them while they are generating, as when their generation is ſuppos'd to be already accompliſh'd, in any other way. A right line, or a curve line, is deſcribed by the motion of a point, a ſurface by the motion of a line, a ſolid by the motion of a ſurface, an angle by the rotation of a radius; all which motions we may conceive to be perform'd according to any ſtated law, as occaſion ſhall require. Theſe generations of quantities we daily ſee to obtain *in rerum naturâ*, and is the manner the ancient Geometricians had often recourſe to, in conſidering their production, and then deducing their properties from ſuch actual deſcriptions. And by analogy, all other quantities, as well as theſe continued geometrical quantities, may be conceived as generated by a kind of motion or progreſs of the Mind.

The Method of Fluxions then ſuppoſes quantities to be generated by local Motion, or ſomething analogous thereto, tho' ſuch generations indeed may not be eſſentially neceſſary to the nature of the thing ſo generated. They might have an exiſtence independent of theſe motions, and may be conceived as produced many other ways, and yet will be endued with the ſame properties. But this conception, of their being now generated by local Motion, is a very fertile notion, and an exceeding uſeful artifice for diſcovering their properties, and a great help to the Mind for a clear, diſtinct, and methodical perception of them. For local Motion ſuppoſes a notion of time, and time implies a ſucceſſion of Ideas. We eaſily diſtinguiſh it into what was, what is, and what will be, in theſe generations of quantities; and ſo we commodiouſly conſider thoſe things by parts, which would be too much for our faculties, and extream difficult for the Mind to take in the whole together, without ſuch artificial partitions and diſtributions.

Our Author therefore makes this eaſy ſuppoſition, that a Line may be conceived as now deſcribing by a Point, which moves either equably or inequably, either with an uniform motion, or elſe according to any rate of continual Acceleration or Retardation. Velocity is a Mathematical Quantity, and like all ſuch, it is ſuſceptible of infinite gradations, may be intended or remitted, may be increaſed or diminiſh'd in different parts of the ſpace deſcribed, according to an infinite variety of ſtated Laws. Now it is plain, that the ſpace thus deſcribed, and the law of acceleration or retardation, (that is, the velocity at every point of time,) muſt have a mutual relation to each other, and muſt mutually determine each other; ſo that one of them being aſſign'd, the other by neceſſary inference may be derived from it. And therefore this is ſtrictly a Geometrical Problem, and capable of a full Determination. And all Geometrical Propoſitions once demonſtrated, or duly inveſtigated, may be ſafely made uſe of, to derive other Propoſitions from them. This will divide the preſent Problem into two Caſes, according as either the Space or Velocity is aſſign'd, at any given time, in order to find the other. And this has given occaſion to that diſtinction which has ſince obtain'd, of the *direct and inverſe Method of Fluxions*, each of which we ſhall now conſider apart.

56. In the direct Method the Problem is thus abſtractedly propoſed. *From the Space deſcribed, being continually given, or aſſumed, or being known at any point of Time aſſign'd; to find the Velocity of the Motion at that Time.* Now in equable Motions it is well known, that the Space deſcribed is always as the Velocity and the Time of deſcription conjunctly; or the Velocity is directly as the Space deſcribed, and reciprocally as the Time of deſcription. And even in inequable Motions, or ſuch as are continually accelerated or retarded, according to ſome ſtated Law, if we take the Spaces and Times very ſmall, they will make a near approach to the nature of equable Motions; and ſtill the nearer, the ſmaller thoſe are taken. But if we may ſuppoſe the Times and Spaces to be indefinitely ſmall, or if they are naſcent or evaneſcent quantities, then we ſhall have the Velocity in any infinitely little Space, as that Space directly, and as the *tempuſculum* inverſely. This property therefore of all inequable Motions being thus deduced, will afford us a medium for ſolving the preſent Problem, as will be ſhewn afterwards. So that the Space deſcribed being thus continually given, and the whole time of its deſcription, the Velocity at the end of that time will be thence determinable.

57. The general abſtract Mechanical Problem, which amounts to the ſame as what is call'd the inverſe Method of Fluxions, will be this. *From the Velocity of the Motion being continually given, to determine the Space deſcribed, at any point of Time aſſign'd.* For the Solution of which we ſhall have the aſſiſtance of this Mechanical Theorem, that in inequable Motions, or when a Point deſcribes a

Line according to any rate of acceleration or retardation, the indefinitely little Space deſcribed in any indefinitely little Time, will be in a compound ratio of the Time and the Velocity ; or the *ſpatiolum* will be as the velocity and the *tempuſculum* conjunctly. This being the Law of all equable Motions, when the Space and Time are any finite quantities, it will obtain alſo in all inequable Motions, when the Space and Time are diminiſh'd *in infinitum*. For by this means all inequable Motions are reduced, as it were, to equability. Hence the Time and the Velocity being continually known, the Space deſcribed may be known alſo; as will more fully appear from what follows. This Problem, in all its caſes, will be capable of a juſt determination; tho' taking it in its full extent, we muſt acknowledge it to be a very difficult and operoſe Problem. So that our Author had good reaſon for calling it *moleſtiſſimum & omnium difficillimum problema*.

58. To fix the Ideas of his Reader, our Author illuſtrates his general Problems by a particular Example. If two Spaces x and y are deſcribed by two points in ſuch manner, that the Space x being uniformly increaſed, in the nature of Time, and its equable velocity being repreſented by the Symbol $\dot{x}$; and if the Space y increaſes inequably, but after ſuch a rate, as that the Equation $y = xx$ ſhall always determine the relation between thoſe Spaces; (or x being continually given, y will be thence known;) then the velocity of the increaſe of y ſhall always be repreſented by $2x\dot{x}$. That is, if the ſymbol $\dot{y}$ be put to repreſent the velocity of the increaſe of y, then will the Equation $\dot{y} = 2x\dot{x}$ always obtain, as will be ſhewn hereafter. Now from the given Equation $y = xx$, or from the relation of the Spaces y and x, (that is, the Space and Time, or its repreſentative,) being continually given, the relation of the Velocities $\dot{y} = 2x\dot{x}$ is found, or the relation of the Velocity $\dot{y}$, by which the Space increaſes, to the Velocity $\dot{x}$, by which the repreſentative of the Time increaſes. And this is an inſtance of the Solution of the firſt general Problem, or of a particular Queſtion in the direct Method of Fluxions. But *vice verſâ*, if the laſt Equation $\dot{y} = 2x\dot{x}$ were given, or if the Velocity $\dot{y}$, by which the Space y is deſcribed, were continually known from the Time x being given, and its Velocity $\dot{x}$; and if from thence we ſhould derive the Equation $y = xx$, or the relation of the Space and Time: This would be an inſtance of the Solution of the ſecond general Problem, or of a particular Queſtion of the inverſe Method of Fluxions. And in analogy to this deſcription of Spaces by moving points, our Author conſiders all other quantities whatever as ge-

nerated and produced by continual augmentation, or by the perpetual acceſſion and accretion of new particles of the ſame kind.

59. In ſettling the Laws of his Calculus of Fluxions, our Author very ſkilfully and judiciouſly diſengages himſelf from all conſideration of Time, as being a thing of too Phyſical or Metaphyſical a nature to be admitted here; eſpecially when there was no abſolute neceſſity for it. For tho' all Motions, and Velocities of Motion, when they come to be compared or meaſured, may ſeem neceſſarily to include a notion of Time; yet Time, like all other quantities, may be repreſented by Lines and Symbols, as in the foregoing example, eſpecially when we conceive them to increaſe uniformly. And theſe repreſentatives or proxies of Time, which in ſome meaſure may be made the objects of Senſe, will anſwer the preſent purpoſe as well as the thing itſelf. So that Time, in ſome ſenſe, may be ſaid to be eliminated and excluded out of the inquiry. By this means the Problem is no longer Phyſical, but becomes much more ſimple and Geometrical, as being wholly confined to the deſcription of Lines and Spaces, with their comparative Velocities of increaſe and decreaſe. Now from the equable Flux of Time, which we conceive to be generated by the continual acceſſion of new particles, or Moments, our Author has thought fit to call his Calculus *the Method of Fluxions.*

60, 61. Here the Author premiſes ſome Definitions, and other neceſſary preliminaries to his Method. Thus Quantities, which in any Problem or Equation are ſuppos'd to be ſuſceptible of continual increaſe or decreaſe, he calls *Fluents*, or *flowing Quantities*; which are ſometimes call'd *variable* or *indeterminate quantities*, becauſe they are capable of receiving an infinite number of particular values, in a regular order of ſucceſſion. The Velocities of the increaſe or decreaſe of ſuch quantities are call'd their *Fluxions*; and quantities in the ſame Problem, not liable to increaſe or decreaſe, or whoſe Fluxions are nothing, are call'd *conſtant*, *given*, *invariable*, and *determinate quantities*. This diſtinction of quantities, when once made, is carefully obſerved through the whole Problem, and inſinuated by proper Symbols. For the firſt Letters of the Alphabet are generally appropriated for denoting conſtant quantities, and the laſt Letters commonly ſignify variable quantities, and the ſame Letters, being pointed, repreſent the Fluxions of thoſe variable quantities or Fluents reſpectively. This diſtinction between theſe quantities is not altogether arbitrary, but has ſome foundation in the nature of the thing, at leaſt during the Solution of the preſent Problem. For the flowing

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or variable quantities may be conceived as *now generating* by Motion, and the conſtant or invariable quantities as ſome how o other *already generated.* Thus in any given Circle or Parabola, the Diameter or Parameter are conſtant lines, or already generated; but the Abſciſs, Ordinate, Area, Curve-line, *&c.* are flowing and variable quantities, becauſe they are to be underſtood as now deſcribing by local Motion, while their properties are derived. Another diſtinction of theſe quantities may be this. A conſtant or given line in any Problem is *linea quædam*, but an indeterminate line is *linea quævis vel quæcunque*, becauſe it may admit of infinite values. Or laſtly, conſtant quantities in a Problem are thoſe, whoſe ratio to a common Unit, of their own kind, is ſuppos'd to be known; but in variable quantities that ratio cannot be known, becauſe it is varying perpetually. This diſtinction of quantities however, into determinate and indeterminate, ſubſiſts no longer than the preſent Calculation requires; for as it is a diſtinction form'd by the Imagination only, for its own conveniency, it has a power of aboliſhing it, and of converting determinate quantities into indeterminate, and *vice verſâ*, as occaſion may require; of which we ſhall ſee Inſtances in what follows. In a Problem, or Equation, there may be any number of conſtant quantities, but there muſt be at leaſt two that are flowing and indeterminate; for one cannot increaſe or diminiſh, while all the reſt continue the ſame. If there are more than two variable quantities in a Problem, their relation ought to be exhibited by more than one Equation.

* * *

ANNOTATIONS on Prob. I.

OR,

The relation of the flowing Quantities being given, to determine the relation of their Fluxions.

SECT. I. *Concerning Fluxions of the firſt order, and to find their Equations.*

1. THE Author having thus propoſed his fundamental Problems, in an abſtract and general manner, and gradually brought them down to the form moſt convenient for his Method; he now proceeds to deliver the Precepts of Solution, which he illuſtrates by a ſufficient variety of Examples; reſerving the Demonſtration to be given afterwards, when his Readers will be better prepared to apprehend the force of it, and when their notions will be better ſettled and confirm'd. Theſe Precepts of Solution, or the Rules for finding the Fluxions of any given Equation, are very ſhort, elegant, and comprehenſive; and appear to have but little affinity with the Rules uſually given for this purpoſe: But that is owing to their great degree of univerſality. We are to form, as it were, ſo many different Tables for the Equation, as there are flowing quantities in it, by diſpoſing the Terms according to the Powers of each quantity, ſo as that their Indices may form an Arithmetical Progreſſion. Then the Terms are to be multiply'd in each caſe, either by the Progreſſion of the Indices, or by the Terms of any other Arithmetical Progreſſion, (which yet ſhould have the ſame common difference with the Progreſſion of the Indices;)

as alſo by the Fluxion of that Fluent, and then to be divided by the Fluent itſelf. Laſt of all, theſe Terms are to be collected, according to their proper Signs, and to be made equal to nothing; which will be a new Equation, exhibiting the relation of the Fluxions. This proceſs indeed is not ſo ſhort as the Method for taking Fluxions, (to be given preſently,) which he elſewhere delivers, and which is commonly follow'd; but it makes ſufficient amends by the univerſality of it, and by the great variety of Solutions which it will afford. For we may derive as many different Fluxional Equations from the ſame given Equation, as we ſhall think fit to aſſume different Arithmetical Progreſſions. Yet all theſe Equations will agree in the main, and tho' differing in form, yet each will truly give the relation of the Fluxions, as will appear from the following Examples.

2. In the firſt Example we are to take the Fluxions of the Equation $x^3 - ax^2 + axy - y^3 = 0$, where the Terms are always brought over to one ſide. Theſe Terms being diſpoſed according to the powers of the Fluent x, or being conſider'd as a Number expreſs'd by the Scale whoſe Root is x, will ſtand thus $x^3 - ax^2 + ayx^1 - y^3x^0 = 0$; and aſſuming the Arithmetical Progreſſion 3, 2, 1, 0, which is here that of the Indices of x, and multiplying each Term by each reſpectively, we ſhall have the Terms $3x^3 - 2ax^2 + ayx *$; which again multiply'd by $\frac{\dot{x}}{x}$, or $\dot{x}x^{-1}$, according to the Rule, will make $3\dot{x}x^2 - 2a\dot{x}x + ay\dot{x}$. Then in the ſame Equation making the other Fluent y the Root of the Scale, it will ſtand thus, $-y^3 + 0y^2 + axy^1 \begin{array}{l} -ax^2 \\ +x^3 \end{array} y^0 = 0$; and aſſuming the Arithmetical Progreſſion 3, 2, 1, 0, which alſo is the Progreſſion of the Indices of y, and multiplying as before, we ſhall have the Terms $-3y^3 * + axy *$, which multiply'd by $\frac{\dot{y}}{y}$, or $\dot{y}y^{-1}$, will make $-3\dot{y}y^2 + ax\dot{y}$. Then collecting the Terms, the Equation $3\dot{x}x^2 - 2a\dot{x}x + ay\dot{x} - 3\dot{y}y^2 + ax\dot{y} = 0$ will give the required relation of the Fluxions. For if we reſolve this Equation into an Analogy, we ſhall have $\dot{x} : \dot{y} :: 3y^2 - ax : 3x^2 - 2ax + ay$; which, in all the values that x and y can aſſume, will give the ratio of their Fluxions, or the comparative velocity of their increaſe or decreaſe, when they flow according to the given Equation.

Or to find this ratio of the Fluxions more immediately, or the value of the Fraction $\frac{\dot{y}}{\dot{x}}$ by fewer ſteps, we may proceed thus. Write down the Fraction $\frac{\dot{y}}{\dot{x}}$ with the note of equality after it, and in the

Numerator of the equivalent Fraction write the Terms of the Equation, diſpos'd according to x, with their reſpective ſigns; each being multiply'd by the Index of x in that Term, (increaſed or diminiſh'd, if you pleaſe, by any common Number,) as alſo divided by x. In the Denominator do the ſame by the Terms, when diſpoſed according to y, only changing the ſigns. Thus in the preſent Equation $x^3 - ax^2 + axy - y^3 = 0$, we ſhall have at once $\frac{\dot{y}}{\dot{x}} = \frac{3x^2 - 2ax + ay \ast}{3y^2 \ast - ax \ast}$.

Let us now apply the Solution another way. The Equation $x^3 - ax^2 + axy - y^3 = 0$ being order'd according to x as before, will be $x^3 - ax^2 + ayx^1 - y^3x^0 = 0$; and ſuppoſing the Indices of x to be increas'd by an unit, or aſſuming the Arithmetical Progreſſion $\frac{4\dot{x}}{x}, \frac{3\dot{x}}{x}, \frac{2\dot{x}}{x}, \frac{\dot{x}}{x}$, and multiplying the Terms reſpectively, we ſhall have theſe Terms $4\dot{x}x^2 - 3a\dot{x}x + 2ay\dot{x} - y^3\dot{x}x^{-1}$. Then ordering the Terms according to y, they will become $-y^3 + 0y^2 + axy^1 + \begin{array}{l} x^3 \\ -ax^2 \end{array} y^0 = 0$; and ſuppoſing the Indices of y to be diminiſh'd by an unit, or aſſuming the Arithmetical Progreſſion $\frac{2\dot{y}}{y}, \frac{\dot{y}}{y}, \frac{c\dot{y}}{y}, \frac{-\dot{y}}{y}$, and multiplying the Terms reſpectively, we ſhall have theſe Terms $-2\dot{y}y^2 \ast \ast - x^3\dot{y}y^{-1} + ax^2\dot{y}y^{-1}$. So that collecting the Terms, we ſhall have $4\dot{x}x^2 - 3a\dot{x}x + 2ay\dot{x} - y^3\dot{x}x^{-1} - 2\dot{y}y^2 - x^3\dot{y}y^{-1} + ax^2\dot{y}y^{-1} = 0$, for the Fluxional Equation required. Or the ratio of the Fluxions will be $\frac{\dot{y}}{\dot{x}} = \frac{4x^2 - 3ax + 2ay - y^3x^{-1}}{2y^2 \ast \ast + x^3y^{-1} - ax^2y^{-1}}$; which ratio may be found immediately by applying the foregoing Rule.

Or contrary-wiſe, if we multiply the Equation in the firſt form by the Progreſſion $\frac{2\dot{x}}{x}, \frac{\dot{x}}{x}, \frac{0\dot{x}}{x}, \frac{-\dot{x}}{x}$, we ſhall have the Terms $2\dot{x}x^2 - a\dot{x}x \ast + y^3\dot{x}x^{-1}$. And if we multiply the Equation in the ſecond form by $\frac{4\dot{y}}{y}, \frac{3\dot{y}}{y}, \frac{2\dot{y}}{y}, \frac{\dot{y}}{y}$, we ſhall have the Terms $-4\dot{y}y^2 \ast + 2ax\dot{y} + x^3\dot{y}y^{-1} - ax^2\dot{y}y^{-1}$. Therefore collecting 'tis $2\dot{x}x^2 - a\dot{x}x + y^3\dot{x}x^{-1} - 4\dot{y}y^2 + 2ax\dot{y} + x^3\dot{y}y^{-1} - ax^2\dot{y}y^{-1} = 0$. Or the ratio of the Fluxions will be $\frac{\dot{y}}{\dot{x}} = \frac{2x^2 - ax \ast + y^3x^{-1}}{4y^2 \ast - 2ax - x^3y^{-1} + ax^2y^{-1}}$, which might have been found at once by the foregoing Rule.

And in general, if the Equation $x^3 - ax^2 + axy - y^3 = 0$, in the form $x^3 - ax^2 + axy - y^3x^0 = 0$, be multiply'd by the Terms of this Arithmetical Progreſſion $\frac{m+3}{x}\dot{x}, \frac{m+2}{x}\dot{x}, \frac{m+1}{x}\dot{x}, \frac{m}{x}\dot{x}$; it will produce the Terms $\overline{m+3}\dot{x}x^2 - \overline{m+2}a\dot{x}x + \overline{m+1}c\dot{x}y - my^3\dot{x}x^{-1}$;

and if the ſame Equation, reduced to the form $-y^3 + 0y^2 + axy^1 + x^3y^0 = 0$, be multiply'd by the Terms of this Arithmetical Pro-
$-ax^2$
greſſion $\frac{n+3}{y}\dot{y}$, $\frac{n+2}{y}\dot{y}$, $\frac{n+1}{y}\dot{y}$, $\frac{n}{y}\dot{y}$, it will produce the Terms $-\overline{n+3}\,\dot{y}y^2$, $*$ $+\overline{n+1}\,ax\dot{y} + nx^3\dot{y}y^{-1} - nax^2\dot{y}y^{-1}$. Then collecting the Terms, we ſhall have $\overline{m+3}\,\dot{x}x^2 - \overline{m+2}\,a\dot{x}x + \overline{m+1}\,a\dot{x}y - my^3\dot{x}x^{-1} - \overline{n+3}\,\dot{y}y^2 \; * + \overline{n+1}\,ax\dot{y} + nx^3\dot{y}y^{-1} - nax^2\dot{y}y^{-1} = 0$, for the Fluxional Equation required. Or the ratio of the Fluxions will be $\frac{\dot{y}}{\dot{x}} = \frac{\overline{m+3}\,x^2 - \overline{m+2}\,ax + \overline{m+1}\,ay - my^3x^{-1}}{\overline{n+3}\,y^2 \; * - \overline{n+1}\,ax - nx^3y^{-1} + nax^2y^{-1}}$; which might have been found immediately from the given Equation, by the foregoing Rule.

Here the general Numbers m and n may be determined *pro lubitu*, by which means we may obtain as many Fluxional Equations as we pleaſe, which will all belong to the given Equation. And thus we may always find the ſimpleſt Expreſſion, or that which is beſt accommodated to the preſent exigence. Thus if we make $m = 0$, and $n = 0$, we ſhall have $\frac{\dot{y}}{\dot{x}} = \frac{3x^2 - 2ax + ay}{3y^2 - ax}$, as found before. Or if we make $m = 1$, and $n = -1$, we ſhall have $\frac{\dot{y}}{\dot{x}} = \frac{4x^2 - 3ax + 2ay - y^3x^{-1}}{2y^2 + x^3y^{-1} - ax^2y^{-1}}$, as before. Or if we make $m = -1$, and $n = 1$, we ſhall have $\frac{\dot{y}}{\dot{x}} = \frac{2x^2 - ax + y^3x^{-1}}{4y^2 - 2ax - x^3y^{-1} + ax^2y^{-1}}$, as before. Or if we make $m = -3$, and $n = -3$, we ſhall have $\frac{\dot{y}}{\dot{x}} = \frac{* \; ax - 2ay + 3y^3x^{-1}}{* \; * \; 2ax + 3x^3y^{-1} - 3ax^2y^{-1}} = \frac{ax^2y - 2axy^2 + 3y^4}{2ax^2y + 3x^4 - 3ax^3}$. And ſo of others. Now this variety of Solutions will beget no ambiguity in the Concluſion, as poſſibly might have been ſuſpected; for it is no other than what ought neceſſarily to ariſe, from the different forms the given Equation may acquire, as will appear afterwards. If we confine ourſelves to the Progreſſion of the Indices, it will bring the Solution to the common Method of taking Fluxions, which our Author has taught elſewhere, and which, becauſe it is eaſy and expeditious, and requires no certain order of the Terms, I ſhall here ſubjoin.

For every Term of the given Equation, ſo many Terms muſt be form'd in the Fluxional Equation, as there are flowing Quantities in that Term. And this muſt be done, (1.) by multiplying the Term by the Index of each flowing Quantity contain'd in it. (2.) By dividing it by the quantity itſelf; and, (3.) by multiplying by its Fluxion. Thus in the foregoing Equation $x^3 - ax^2 + ayx - y^3 = 0$, the Fluxion belonging to the Term x^3 is $\frac{3x^3\dot{x}}{x}$, or $3x^2\dot{x}$.

The Fluxion belonging to $-ax^2$ is $-\frac{2ax^2\dot{x}}{x}$, or $-2ax\dot{x}$. The Fluxion belonging to ayx is $\frac{ayx\dot{y}}{y}+\frac{ayx\dot{x}}{x}$, or $ax\dot{y}+ay\dot{x}$. And the Fluxion belonging to $-y^3$ is $-\frac{3y^3\dot{y}}{y}$, or $-3y^2\dot{y}$. So that the Fluxion of the whole Equation, or the whole Fluxional Equation, is $3x^2\dot{x}-2ax\dot{x}+a\dot{y}x+ay\dot{x}-3y^2\dot{y}=0$. Thus the Equation $x^m=y$, will give $m\dot{x}x^{m-1}=\dot{y}$; and the Equation $x^mz^n=y$, will give $m\dot{x}x^{m-1}z^n+nx^m\dot{z}z^{n-1}=\dot{y}$ for its Fluxional Equation. And the like of other Examples.

If we take the Author's ſimple Example, in pag. 19, or the Equation $y=xx$, or rather $ay-x^2=0$, that is $ayx^0-x^2y^0=0$, in order to find its moſt general Fluxional Equation; it may be perform'd by the Rule before given, ſuppoſing the Index of x to be encreas'd by m, and the Index of y by n. For then we ſhall have directly $\frac{\dot{y}}{\dot{x}}=\frac{mayx^{-1}-\overline{m+2}x}{nx^2y^{-1}-\overline{n+1}a}$. For the firſt Term of the given Equation being ayx^0, this multiply'd by the Index of x increas'd by m, that is by m, and divided by x, will give $mayx^{-1}$ for the firſt Term of the Numerator. Alſo the ſecond Term being $-x^2y^0$, this multiply'd by the Index of x increas'd by m, that is by $m+2$, and divided by x, will give $-\overline{m+2}x$ for the ſecond Term of the Numerator. Again, the firſt Term of the given Equation may be now $-x^2y^0$, which multiply'd by the Index of y increas'd by n, that is by n, and divided by y, will give (changing the ſign) nx^2y^{-1} for the firſt Term of the Denominator. Alſo the ſecond Term will then be ayx^0, which multiply'd by the Index of y increas'd by n, that is by $n+1$, and divided by y, will give (changing the Sign) $-\overline{n+1}a$ for the ſecond Term of the Denominator, as found above. Now from this general relation of the Fluxions, we may deduce as many particular ones as we pleaſe. Thus if we make $m=0$, and $n=0$, we ſhall have $\frac{\dot{y}}{\dot{x}}=\frac{2x}{a}$, or $a\dot{y}=2x\dot{x}$, agreeable to our Author's Solution in the place before cited. Or if we make $m=-2$, and $n=-1$, we ſhall have $\frac{\dot{y}}{\dot{x}}=\frac{2ayx^{-1}}{x^2y^{-1}}=\frac{2ay^2}{x^3}$. Or if we make $m=0$, and $n=-1$, we ſhall have $\frac{\dot{y}}{\dot{x}}=\frac{2x}{x^2y^{-1}}=\frac{2y}{x}$. Or if we make $n=0$, and $m=-2$, we ſhall have $\frac{\dot{y}}{\dot{x}}=\frac{2ayx^{-1}}{a}=\frac{2y}{x}$, as before. All which, and innumerable other caſes, may be eaſily proved by a ſubſtitution of equivalents. Or we may prove it generally thus. Becauſe by the given Equation it is $y = x^2a^{-1}$, in the value of the ratio $\frac{\dot{y}}{\dot{x}} = \frac{mayx^{-1} - \overline{m+2}x}{nx^2y^{-1} - \overline{n+1}a}$ for y ſubſtitute its value, and we ſhall have $\frac{\dot{y}}{\dot{x}} = \frac{mx - \overline{m+2}x}{na - \overline{n+1}a} = \frac{2x}{a}$, as above.

3. The Equation of the ſecond Example is $2y^3 + x^2y - 2cyz + 3yz^2 - z^3 = 0$, in which there are three flowing quantities y, x, and z, and therefore there muſt be three operations, or three Tables muſt be form'd. Firſt diſpoſe the Terms according to y, thus;

$$2y^3 + 0y^2 \begin{array}{l} +\ x^2 \\ -\ 2cz \\ +\ 3z^2 \end{array} y - z^3y^0 = 0,$$

and multiply by the Terms of the Progreſſion $2 \times \dot{y}y^{-1}$, $1 \times \dot{y}y^{-1}$, $0 \times \dot{y}y^{-1}$, $-1 \times \dot{y}y^{-1}$, reſpectively, (where the Coefficients are form'd by diminiſhing the Indices of y by the common Number 1,) and the reſulting Terms will be $4\dot{y}y^2 \ast \ast + z^3\dot{y}y^{-1}$. Secondly, diſpoſe the Terms according to x, thus;

$$yx^2 + 0x \begin{array}{l} +\ 2y^3 \\ -\ 2cyz \\ +\ 3yz^2 \\ -\ z^3 \end{array} x^0 = 0,$$

and multiply by the Terms of the Progreſſion $2 \times \dot{x}x^{-1}$, $1 \times \dot{x}x^{-1}$, $0 \times \dot{x}x^{-1}$, (where the Coefficients are the ſame as the Indices of x,) and the only reſulting Term here is $+ 2y\dot{x}x \ast \ast$. Laſtly, diſpoſe the Terms according to z, thus;

$$-z^3 + 3yz^2 - 2cyz \begin{array}{l} +\ x^2y \\ +\ 2y^3 \end{array} z^0 = 0,$$

and multiply by the Progreſſion $3 \times \dot{z}z^{-1}$, $2 \times \dot{z}z^{-1}$, $1 \times \dot{z}z^{-1}$, $0 \times \dot{z}z^{-1}$, (where the Coefficients are alſo the ſame as the Indices of z,) and the Terms will be $-3\dot{z}z^2 + 6y\dot{z}z - 2cy\dot{z} \ast$. Then collecting all theſe Terms together, we ſhall have the Fluxional Equation $4\dot{y}y^2 + z^3\dot{y}y^{-1} + 2y\dot{x}x - 3\dot{z}z^2 + 6y\dot{z}z - 2cy\dot{z} = 0$.

Here we have a notable inſtance of our Author's dexterity, at finding expedients for abbreviating. For in every one of theſe Operations ſuch a Progreſſion is choſe, as by multiplication will make the greateſt deſtruction of the Terms. By which means he arrives at the ſhorteſt Expreſſion, that the nature of the Problem will allow. If we ſhould ſeek the Fluxions of this Equation by the uſual method, which is taught above, that is, if we always aſſume the Progreſſions of the Indices, we ſhall have $6\dot{y}y^2 + 2\dot{x}xy + x^2\dot{y} - 2c\dot{y}z - 2cy\dot{z} + 3\dot{y}z^2 + 6yz\dot{z} - 3\dot{z}z^2 = 0$; which has two Terms more than the other form. And if the Progreſſions of the Indices are increas'd, in each caſe, by any common general Numbers, we may form the moſt general Expreſſion for the Fluxional Equation, that the Problem will admit of.

4. On occaſion of the laſt Example, in which are three Fluents and their Fluxions, our Author makes an uſeful Obſervation, for the Reduction and compleat Determination of ſuch Equations, tho' it be derived from the Rules of the vulgar Algebra ; which matter may be conſider'd thus. Every Equation, conſiſting of two flowing or variable Quantities, is what correſponds to an indetermin'd Problem, admitting of an infinite number of Anſwers. Therefore one of thoſe quantities being aſſumed at pleaſure, or a particular value being aſſign'd to it, the other will alſo be compleatly determined. And in the Fluxional Equation derived from thence, thoſe particular values being ſubſtituted, the Ratio of the Fluxions will be given in Numbers, in any particular caſe. And one of the Fluxions being taken for Unity, or of any determinate value, the value of the other may be exhibited by a Number, which will be a compleat Determination.

But if the given Equation involve three flowing or indeterminate Quantities, two of them muſt be aſſumed to determine the third ; or, which is the ſame thing, ſome other Equation muſt be either given or aſſumed, involving ſome or all the Fluents, in order to a compleat Determination. For then, by means of the two Equations, one of the Fluents may be eliminated, which will bring this to the former caſe. Alſo two Fluxional Equations may be derived, involving the three Fluxions, by means of which one of them may be eliminated. And ſo if the given Equation ſhould involve four Fluents, two other Equations ſhould be either given or aſſumed, in order to a compleat Determination. This will be ſufficiently explain'd by the two following Examples, which will alſo teach us how complicate Terms, ſuch as compound Fractions and Surds, are to be managed in this Method.

5, 6. Let the given Equation be $y^2 - a^2 - x\sqrt{a^2 - x^2} = 0$, of which we are to take the Fluxions. To the two Fluents y and x we may introduce a third z, if we aſſume another Equation. Let that be $z = x\sqrt{a^2 - x^2}$, and we ſhall have the two Equations $y^2 - a^2 - z = 0$, and $a^2x^2 - x^4 - z^2 = 0$. Then by the foregoing Solution their Fluxional Equations (at leaſt in one caſe) will be $2\dot{y}y - \dot{z} = 0$, and $a^2\dot{x}x - 2\dot{x}x^3 - \dot{z}z = 0$. Theſe two Fluential Equations, and their Fluxional Equations, may be reduced to one Fluential and one Fluxional Equation, by the uſual methods of Reduction : that is, we may eliminate z and $\dot{z}$ by ſubſtituting their values $yy - aa$ and $2\dot{y}y$. Then we ſhall have $y^2 - a^2 - x\sqrt{a^2 - x^2}$ $= 0$,

$= 0$, and $2\dot{y}y - \frac{a^2\dot{x} - 2\dot{x}x^3}{\sqrt{a^2 - x^2}} = 0$. Or by taking away the ſurds, 'tis $a^2x^2 - x^4 - y^4 + 2a^2y^2 - a^4 = 0$, and then $a^2\dot{x}x - 2\dot{x}x^3 - 2\dot{y}y^3 + 2a^2\dot{y}y = 0$.

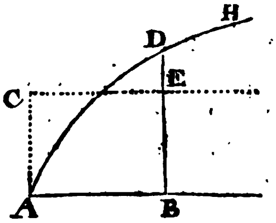
7. Or if the given Equation be $x^3 - ay^2 + \frac{by^3}{a+y} - x^2\sqrt{ay + x^2} = 0$, to find its correſponding Fluxional Equation; to the two flowing quantities x and y we may introduce two others z and v, and thereby remove the Fraction and the Radical, if we aſſume the two Equations $\frac{by^3}{a+y} = z$, and $x^2\sqrt{ay + xx} = v$. Then we ſhall have the three Equations $x^3 - ay^2 + z - v = 0$, $az + yz - by^3 = 0$, and $ayx^4 + x^6 - v^2 = 0$, which will give the three Fluxional Equations $3\dot{x}x^2 - 2a\dot{y}y + \dot{z} - \dot{v} = 0$, $a\dot{z} + y\dot{z} + \dot{y}z - 3b\dot{y}y^2 = 0$, and $a\dot{y}x^4 + 4ay\dot{x}x^3 + 6\dot{x}x^5 - 2\dot{v}v = 0$. Theſe by known Methods of the common Algebra may be reduced to one Fluential and one Fluxional Equation, involving x and y, and their Fluxions, as is required.

8. And by the ſame Method we may take the Fluxions of Binomial or other Radicals, of any kind, any how involved or complicated with one another. As for inſtance, if we were to find the Fluxion of $\sqrt{ax + \sqrt{aa - xx}}$, put it equal to y, or make $ax + \sqrt{aa - xx} = yy$. Alſo make $\sqrt{aa - xx} = z$. Then we ſhall have the two Fluential Equations $ax + z - y^2 = 0$, and $a^2 - x^2 - z^2 = 0$, from whence we ſhall have the two Fluxional Equations $a\dot{x} + \dot{z} - 2\dot{y}y = 0$; and $-2\dot{x}x - 2\dot{z}z = 0$, or $\dot{x}x + \dot{z}z = 0$. This laſt Equation, if for z and $\dot{z}$ we ſubſtitute their values $yy - ax$ and $2\dot{y}y - a\dot{x}$, will become $x\dot{x} + 2\dot{y}y^3 - 2ax\dot{y}y - a\dot{x}y^2 + a^2\dot{x}x = 0$; whence $\dot{y} = \frac{a\dot{x}y^2 - a^2\dot{x}x - \dot{x}x}{2y^3 - 2ax}$. And here if for y we ſubſtitute its value $\sqrt{ax + \sqrt{aa - xx}}$, we ſhall have the Fluxion required $\dot{y} = \frac{a\dot{x}\sqrt{aa - xx} - \dot{x}x}{2\sqrt{aa - xx} \times \sqrt{ax + \sqrt{aa - xx}}}$. And many other Examples of a like kind will be found in the ſequel of this Work.

9, 10, 11, 12. In Examp. 5. the propoſed Equation is $zz + axz - y^4 = 0$, in which there are three variable quantities x, y, and z, and therefore the relation of the Fluxions will be $2\dot{z}z + a\dot{x}z + ax\dot{z} - 4\dot{y}y^3 = 0$. But as there wants another Fluential Equation, and thence another Fluxional Equation, to make a compleat determination; if only another Fluxional Equation were given or aſſumed, we ſhould have the required relation of the Fluxions $\dot{x}$ and $\dot{y}$.

Suppoſe this Fluxional Equation were $\dot{z}=\dot{x}\sqrt{ax-xx}$; then by ſubſtitution we ſhould have the Equation $\overline{2z+ax}\times\dot{x}\sqrt{ax-xx}+a\dot{x}z-4\dot{y}y^3=0$, or the Analogy $\dot{x}:\dot{y}::4y^3:\overline{2z+ax}\times\sqrt{ax-xx}+az$, which can be reduced no farther, (or z cannot be eliminated,) till we have the Fluential Equation, from which the Fluxional Equation $\dot{z}=\dot{x}\sqrt{ax-xx}$ is ſuppos'd to be derived. And thus we may have the relation of the Fluxions, even in ſuch caſes as we have not, or perhaps cannot have, the relation of the Fluents.

But tho' this Reduction may not perhaps be conveniently perform'd Analytically, or by Calculation, yet it may poſſibly be perform'd Geometrically, as it were, and by the Quadrature of Curves; as we may learn from our Author's preparatory Propoſition, and from the following general Conſiderations. Let the right Line AC, perpendicular to the right Line AB, be conceived to move always parallel to itſelf, ſo as that its extremity A may deſcribe the line AB. Let the point C be fixt, or always at the ſame diſtance from A, and let another point move from A towards C, with a velocity any how accelerated or retarded. The parallel motion of the line AC does not at all affect the progreſſive motion of the point moving from A towards C, but from a combination of theſe two independent motions, it will deſcribe the Curve ADH; while at the ſame time the fixt point C will deſcribe the right line CE, parallel to AB. Let the line AC be conceived to move thus, till it comes into the place BE, or BD. Then the line AC is conſtant, and remains the ſame, while the indefinite or flowing line becomes BD. Alſo the Areas deſcribed at the ſame time, ACEB and ADB, are likewiſe flowing quantities, and their velocities of deſcription, or their Fluxions, muſt neceſſarily be as their reſpective deſcribing lines, or Ordinates, BE and BD. Let AC or BE be Linear Unity, or a conſtant known right line, to which all the other lines are to be compared or refer'd; juſt as in Numbers, all other Numbers are tacitely refer'd to 1, or to Numeral Unity, as being the ſimpleſt of all Numbers. And let the Area ADB be ſuppos'd to be apply'd to BE, or Linear Unity, by which it will be reduced from the order of Surfaces to that of Lines; and let the reſulting line be call'd z. That is, make the Area ADB $=z\times$ BE; and if AB be call'd x, then is the Area ACEB $=x\times$ BE. Therefore the



Fluxions of theſe Areas will be $\dot{z} \times BE$ and $\dot{x} \times BE$, which are as $\dot{z}$ and $\dot{x}$. But the Fluxions of the Areas were found before to be as BD to BE. So that it is $\dot{z} : \dot{x} :: BD : BE = 1$, or $\dot{z} = \dot{x} \times BD$. Conſequently in any Curve, the Fluxion of the Area will be as the Ordinate of the Curve, drawn into the Fluxion of the Abſciſs.

Now to apply this to the preſent caſe. In the Fluxional Equation before aſſumed $\dot{z} = \dot{x}\sqrt{ax - xx}$, if x repreſents the Abſciſs of a Curve, and $\sqrt{ax - xx}$ be the Ordinate; then will this Curve be a Circle, and z will repreſent the correſponding Area. So that we ſee from hence, whether the Area of a Circle can be exhibited or no, or, in general Terms, tho' in the Equation propoſed there ſhould be quantities involved, which cannot be determined or expreſs'd by any Geometrical Method; ſuch as the Areas or Lengths of Curve-lines; yet the relation of their Fluxions may nevertheleſs be found.

13. We now come to the Author's Demonſtration of his Solution, or to the proof of the Principles of the Method of Fluxions, here laid down; which certainly deſerves to engage our moſt ſerious attention. And more eſpecially, becauſe theſe Principles have been lately drawn into debate, without being well conſider'd or underſtood; poſſibly becauſe this Treatiſe of our Author's, expreſly wrote on the ſubject, had not yet ſeen the light. As theſe Principles therefore have been treated as precarious at leaſt, if not wholly inſufficient to ſupport the Doctrine derived from them; I ſhall endeavour to examine into every, the moſt minute circumſtance of this Demonſtration, and that with the utmoſt circumſpection and impartiality.

We have here in the firſt place a Definition and a Theorem together. *Moments* are defined to be *the indefinitely ſmall parts of flowing quantities, by the acceſſion of which, in indefinitely ſmall portions of time, theſe quantities are continually increaſed.* The word *Moment* (*momentum, movimentum, à moveo,*) by analogy ſeems to have been borrow'd from Time. For as Time is conceived to be in continual flux, or motion, and as a greater and a greater Time is generated by the acceſſion of more and more Moments, which are conceived as the ſmalleſt particles of Time: So all other flowing Quantities may be underſtood as perpetually increaſing, by the acceſſion of their ſmalleſt particles, which therefore may not improperly be call'd their Moments. But what are here call'd their *ſmalleſt particles*, are not to be underſtood as if they were Atoms, or of any definite and determinate magnitude, as in the Method of Indiviſibles; but to be indefinitely ſmall, or continually decreaſing, till they are leſs than any aſſignable quantities, and yet may then retain all poſſible varieties of proportion to one another. That theſe Moments are not chimerical, viſionary, or merely imaginary things, but have an exiſtence *ſui generis*, at leaſt Mathematically and in the Underſtanding, is a neceſſary conſequence from the infinite Diviſibility of Quantity, which I think hardly any body now conteſts [*]. For all continued quantity whatever, tho' not indeed actually, yet mentally may be conceived to be divided *in infinitum*. Perhaps this may be beſt illuſtrated by a comparative gradation or progreſs of Magnitudes. Every finite and limited *Quantity* may be conceived as divided into any finite number of ſmaller *parts*. This Diviſion may proceed, and thoſe parts may be conceived to be farther divided in very little, but ſtill finite parts, or *particles*, which yet are not *Moments*. But when theſe particles are farther conceived to be divided, not actually but mentally, ſo far as to become of a magnitude leſs than any aſſignable, (and what can ſtop the progreſs of the Mind?) then are they properly the *Moments* which are to be underſtood here. As this gradation of diminution certainly includes no abſurdity or contradiction, the Mind has the privilege of forming a Conception of theſe Moments, a poſſible Notion at leaſt, though perhaps not an adequate one; and then Mathematicians have a right of applying them to uſe, and of making ſuch Inferences from them, as by any ſtrict way of reaſoning may be derived.

It is objected, that we cannot form an intelligible and adequate Notion of theſe Moments, becauſe ſo obſcure and incomprehenſible an Idea, as that of Infinity is, muſt needs enter that Notion; and therefore they ought to be excluded from all Geometrical Diſquiſitions. It may indeed be allowed, that we have not an adequate Notion of them on that account, ſuch as exhauſts the whole nature of the thing, neither is it at all neceſſary; for a partial Notion, which is that of their Diviſibility *ſine fine*, without any regard to their magnitude, is ſufficient in the preſent caſe. There are many other Speculations in the Mathematicks, in which a Notion of Infinity is a neceſſary ingredient, which however are admitted by all Geometricians, as uſeful and demonſtrable Truths. The Doctrine of commenſurable and incommenſurable magnitudes includes a Notion of Infinity, and yet is received as a very demonſtrable Doctrine. We have a perfect Idea of a Square and its Diagonal, and yet we know

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[*] Perhaps the ingenious Author of the Diſcourſe call'd *The Analyſt* muſt be excepted, who is pleas'd to aſk, in his fifth Query, *Whether it be not unneceſſary, as well as abſurd, to ſuppoſe that finite Extenſion is infinitely diviſible?* See alſo Query 19, 20, 21, &c.

know they will admit of no finite common meaſure, or that their proportion cannot be exhibited in rational Numbers, tho' ever ſo ſmall, but may by a ſeries of decimal or other parts continued *ad infinitum.* In common Arithmetick we know, that the vulgar Fraction $\frac{2}{3}$, and the decimal Fraction 0,666666, &c. continued *ad infinitum*, are one and the ſame thing; and therefore if we have a ſcientifick notion of the one, we have likewiſe of the other. When I deſcribe a right line with my Pen, ſuppoſe of an Inch long, I deſcribe firſt one half of the line, then one half of the remainder, then one half of the next remainder, and ſo on. That is, I actually run over all thoſe infinite diviſions and ſubdiviſions, before I have compleated the Line, tho' I do not attend to them, or cannot diſtinguiſh them. And by this I am indubitably certain, that this Series of Fractions $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, &c. continued *ad infinitum*, is preciſely equal to Unity. *Euclid* has demonſtrated in his Elements, that the Circular Angle of Contact is leſs than any aſſignable right-lined Angle, or, which is the ſame thing, is an infinitely little Angle in compariſon with any finite Angle: And our Author ſhews us ſtill greater Myſteries, about the infinite gradations of Angles of Contact. In Geometry we know, that Curves may continually approach towards their Aſymptotes, and yet will not actually meet with them, till both are continued to an infinite diſtance. We know likewiſe, that many of their included Areas, or Solids, will be but of a finite and determinable magnitude, even tho' their lengths ſhould be actually continued *ad infinitum*. We know that ſome Spirals make infinite Circumvolutions about a Pole, or Center, and yet the whole Line, thus infinitely involved, is but of a finite, determinable, and aſſignable length. The Methods of computing Logarithms ſuppoſe, that between any two given Numbers, an infinite number of mean Proportionals may be interpoſed; and without ſome Notion of Infinity their nature and properties are hardly intelligible or diſcoverable. And in general, many of the moſt ſublime and uſeful parts of knowledge muſt be baniſh'd out of the Mathematicks, if we are ſo ſcrupulous as to admit of no Speculations, in which a Notion of Infinity will be neceſſarily included. We may therefore as ſafely admit of Moments, and the Principles upon which the Method of Fluxions is here built, as any of the fore-mention'd Speculations.

The nature and notion of Moments being thus eſtabliſh'd, we may paſs on to the afore-mention'd Theorem, which is this.

The (contemporary) Moments of flowing quantities are as the Velocities of flowing or increaſing; that is, as their Fluxions. Now if this be proved of Lines, it will equally obtain in all flowing quantities whatever, which may always be adequately repreſented and expounded by Lines. But in equable Motions, the Times being given, the Spaces deſcribed will be as the Velocities of Deſcription, as is known in Mechanicks. And if this be true of any finite Spaces whatever, or of all Spaces in general, it muſt alſo obtain in infinitely little Spaces, which we call Moments. And even in Motions continually accelerated or retarded, the Motions in infinitely little Spaces, or Moments, muſt degenerate into equability. So that the Velocities of increaſe or decreaſe, or the Fluxions, will be always as the contemporary Moments. Therefore the Ratio of the Fluxions of Quantities, and the Ratio of their contemporary Moments, will always be the ſame, and may be uſed promiſcuouſly for each other.

14. The next thing to be ſettled is a convenient Notation for theſe Moments, by which they may be diſtinguiſh'd, repreſented, compared, and readily ſuggeſted to the Imagination. It has been appointed already, that when x, y, z, v, &c. ſtand for variable or flowing quantities, then their Velocities of increaſe, or their Fluxions, ſhall be repreſented by $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{v}$, &c. which therefore will be proportional to the contemporary Moments. But as theſe are only Velocities, or magnitudes of another Species, they cannot be the Moments themſelves, which we conceive as indefinitely little Spaces, or other analogous quantities. We may therefore here aptly introduce the Symbol o, not to ſtand for abſolute nothing, as in Arithmetick, but a vaniſhing Space or Quantity, which was juſt now finite, but by continually decreaſing, in order preſently to terminate in mere nothing, is now become leſs than any aſſignable Quantity. And we have certainly a right ſo to do. For if the notion is intelligible, and implies no contradiction as was argued before, it may ſurely be inſinuated by a Character appropriate to it. This is not aſſigning the quantity, which would be contrary to the *hypotheſis*, but is only appointing a mark to repreſent it. Then multiplying the Fluxions by the vaniſhing quantity o, we ſhall have the ſeveral quantities $\dot{x}o$, $\dot{y}o$, $\dot{z}o$, $\dot{v}o$, &c. which are vaniſhing likewiſe, and proportional to the Fluxions reſpectively. Theſe therefore may now repreſent the contemporary Moments of x, y, z, v, &c. And in general, whatever other flowing quantities, as well as Lines and

Spaces, are repreſented by x, y, z, v, &c. as o may ſtand for a vaniſhing quantity of the ſame kind, and as $\dot{x}$, $\dot{y}$, $\dot{z}$, $\dot{v}$, &c. may ſtand for their Velocities of increaſe or decreaſe, (or, if you pleaſe, for Numbers proportional to thoſe Velocities,) then may $\dot{x}o$, $\dot{y}o$, $\dot{z}o$, $\dot{v}o$, &c. always denote their reſpective ſynchronal Moments, or momentary acceſſions, and may be admitted into Computations accordingly. And this we come now to apply.

15. We muſt now have recourſe to a very notable, uſeful, and extenſive property, belonging to all Equations that involve flowing Quantities. Which property is, that in the progreſs of flowing, the Fluents will continually acquire new values, by the acceſſion of contemporary parts of thoſe Fluents, and yet the Equation will be equally true in all theſe caſes. This is a neceſſary reſult from the Nature and Definition of variable Quantities. Conſequently theſe Fluents may be any how increaſed or diminiſh'd by their contemporary Increments or Decrements; which Fluents, ſo increaſed or diminiſhed, may be ſubſtituted for the others in the Equation. As if an Equation ſhould involve the Fluents x and y, together with any given quantities, and X and Y are ſuppoſed to be any of their contemporary Augments reſpectively. Then in the given Equation we may ſubſtitute $x+X$ for x, and $y+Y$ for y, and yet the Equation will be good, or the equality of the Terms will be preſerved. So if X and Y were contemporary Decrements, inſtead of x and y we might ſubſtitute $x-X$ and $y-Y$ reſpectively. And as this muſt hold good of all contemporary Increments or Decrements whatever, whether finitely great or infinitely little, it will be true likewiſe of contemporary Moments. That is, inſtead of x and y in any Equation, we may ſubſtitute $x+\dot{x}o$ and $y+\dot{y}o$, and yet we ſhall ſtill have a good Equation. The tendency of this will appear from what immediately follows.

16. The Author's ſingle Example is a kind of Induction, and the proof of this may ſerve for all caſes. Let the Equation $x^3-ax^2+axy-y^3=0$ be given as before, including the variable quantities x and y, inſtead of which we may ſubſtitute theſe quantities increas'd by their contemporary Moments, or $x+\dot{x}o$ and $y+\dot{y}o$ reſpectively. Then we ſhall have the Equation $\overline{x+\dot{x}o}|^3 - a\times\overline{x+\dot{x}o}|^2 + a\times\overline{x+\dot{x}o}\times\overline{y+\dot{y}o} - \overline{y+\dot{y}o}|^3 = 0$. Theſe Terms being expanded, and reduced to three orders or columns, according as the vaniſhing quantity o is of none, one, or of more dimenſions, will ſtand as in the Margin.

$$\left.\begin{array}{lll} x^3 + 3\dot{x}ox^2 & + 3\dot{x}^2o^2x \\ & + \dot{x}^3o^3 \\ -ax^2 - 2a\dot{x}ox & - a\dot{x}^2o^2 \\ +axy + a\dot{x}oy & + a\dot{x}\dot{y}o^2 \\ \quad\;\; + a\dot{y}ox & \\ -y^3 - 3\dot{y}oy^2 & - 3\dot{y}^2o^2y \\ & - \dot{y}^3o^3 \end{array}\right\} = 0.$$

17, 18. Here the Terms of the firſt order, or column, remove or deſtroy one another, as being abſolutely equal to nothing by the given Equation. They being therefore expunged, the remaining Terms may all be divided by the common Multiplier o, whatever it is. This being done, all the Terms of the third order will ſtill be affected by o, of one or more dimenſions, and may therefore be expunged, as infinitely leſs than the others. Laſtly, there will only remain thoſe of the ſecond order or column, that is $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$, which will be the Fluxional Equation required. Q. E. D.

The ſame Concluſions may be thus derived, in ſomething a different manner. Let X and Y be any ſynchronal Augments of the variable quantities x and y, as before, the relation of which quantities is exhibited by any Equation. Then may $x + \mathrm{X}$ and $y + \mathrm{Y}$ be ſubſtituted for x and y in that Equation. Suppoſe for inſtance that $x^3 - ax^2 + axy - y^3 = 0$; then by ſubſtitution we ſhall have $\overline{x + \mathrm{X}}|^3 - a \times \overline{x + \mathrm{X}}|^2 + a \times \overline{x + \mathrm{X}} \times \overline{y + \mathrm{Y}} - \overline{y + \mathrm{Y}}|^3 = 0$; or *in terminis expanſis* $x^3 + 3x^2\mathrm{X} + 3x\mathrm{X}^2 + \mathrm{X}^3 - ax^2 - 2ax\mathrm{X} - a\mathrm{X}^2 + axy + ax\mathrm{Y} + a\mathrm{X}y + a\mathrm{XY} - y^3 - 3y^2\mathrm{Y} - 3y\mathrm{Y}^2 - \mathrm{Y}^3 = 0$. But the Terms $x^3 - ax^2 + axy - y^3 = 0$ will vaniſh out of the Equation, and leave $3x^2\mathrm{X} + 3x\mathrm{X}^2 + \mathrm{X}^3 - 2ax\mathrm{X} - a\mathrm{X}^2 + ax\mathrm{Y} + a\mathrm{X}y + a\mathrm{XY} - 3y^2\mathrm{Y} - 3y\mathrm{Y}^2 - \mathrm{Y}^3 = 0$, for the relation of the contemporary Augments, let their magnitude be what it will. Or reſolving this Equation into an Analogy, the ratio of theſe Augments may be this, $\frac{\mathrm{Y}}{\mathrm{X}} = \frac{3x^2 + 3x\mathrm{X} + \mathrm{X}^2 - 2ax - a\mathrm{X} + ay}{-ax - a\mathrm{X} + 3y^2 + 3y\mathrm{Y} + \mathrm{Y}^2}$. Now to find the *ultimate ratio* of theſe Augments, or their ratio when they become Moments, ſuppoſe X and Y to diminiſh till they become vaniſhing quantities, and then they may be expunged out of this value of the ratio. Or in thoſe circumſtances it will be $\frac{\mathrm{Y}}{\mathrm{X}} = \frac{3x^2 - 2ax + ay}{3y^2 - ax}$, which is now the ratio of the Moments. And this is the ſame ratio as that of the Fluxions, or it will be $\frac{\dot{y}}{\dot{x}} = \frac{3x^2 - 2ax + ay}{3y^2 - ax}$, or $3y^2\dot{y} - ax\dot{y} = 3x^2\dot{x} - 2ax\dot{x} + ay\dot{x}$, as was found before.

In this way of arguing there is no aſſumption made, but what is juſtifiable by the received Methods both of the ancient and modern Geometricians. We only deſcend from a general Propoſition, which is undeniable, to a particular caſe which is certainly included in it.

That is, having the relation of the variable Quantities, we thence directly deduce the relation or ratio of their contemporary Augments; and having this, we directly deduce the relation or ratio of thoſe contemporary Augments when they are naſcent or evaneſcent, juſt beginning or juſt ceaſing to be; in a word, when they are Moments, or vaniſhing Quantities. To evade this reaſoning, it ought to be proved, that no Quantities can be conceived leſs than aſſignable Quantities; that the Mind has not the privilege of conceiving Quantity as perpetually diminiſhing *ſine fine*; that the Conception of a vaniſhing Quantity, a Moment, an Infiniteſimal, &c. includes a contradiction: In ſhort, that Quantity is not (even mentally) diviſible *ad infinitum*; for to that the Controverſy muſt be reduced at laſt. But I believe it will be a very difficult matter to extort this Principle from the Mathematicians of our days, who have been ſo long in quiet poſſeſſion of it, who are indubitably convinced of the evidence and certainty of it, who continually and ſucceſsfully apply it, and who are ready to acknowledge the extreme fertility and uſefulneſs of it, upon ſo many important occaſions.

19. Nothing remains, I think, but to account for theſe two circumſtances, belonging to the Method of Fluxions, which our Author briefly mentions here. Firſt that the given Equation, whoſe Fluxional Equation is to be found, may involve any number of flowing quantities. This has been ſufficiently proved already, and we have ſeen ſeveral Examples of it. Secondly, that in taking Fluxions we need not always confine ourſelves to the progreſſion of the Indices, but may aſſume infinite other Arithmetical Progreſſions, as conveniency may require. This will deſerve a little farther illuſtration, tho' it is no other than what muſt neceſſarily reſult from the different forms, which any given Equation may aſſume, in an infinite variety. Thus the Equation $x^3 - ax^2 + axy - y^3 = 0$, being multiply'd by the general quantity $x^m y^n$, will become $x^{m+3}y^n - ax^{m+2}y^n + ax^{m+1}y^{n+1} - x^m y^{n+3} = 0$, which is virtually the ſame Equation as it was before, tho' it may aſſume infinite forms, according as we pleaſe to interpret m and n. And if we take the Fluxions of this Equation, in the uſual way, we ſhall have $\overline{m+3}\dot{x}x^{m+2}y^n + nx^{m+3}\dot{y}y^{n-1} - \overline{m+2}a\dot{x}x^{m+1}y^n - nax^{m+2}\dot{y}y^{n-1} + \overline{m+1}a\dot{x}x^m y^{n+1} + \overline{n+1}ax^{m+1}\dot{y}y^n - m\dot{x}x^{m-1}y^{n+3} - \overline{n+3}x^m\dot{y}y^{n+2} = 0$. Now if we divide this again by $x^m y^n$, we ſhall have $\overline{m+3}\dot{x}x^2 + nx^3\dot{y}y^{-1} - \overline{m+2}a\dot{x}x - nax^2\dot{y}y^{-1} + \overline{m+1}a\dot{x}y + \overline{n+1}ax\dot{y} - m\dot{x}x^{-1}y^3 - \overline{n+3}\dot{y}y^2 = 0$, which is the ſame general Equation as was derived before. And the like may be underſtood of all other Examples.

SECT.

SECT. II. *Concerning Fluxions of ſuperior orders, and the method of deriving their Equations.*

IN this Treatiſe our Author conſiders only firſt Fluxions, and has not thought fit to extend his Method to ſuperior orders, as not directly falling within his preſent purpoſe. For tho' he here purſues Speculations which require the uſe of ſecond Fluxions, or higher orders, yet he has very artfully contrived to reduce them to firſt Fluxions, and to avoid the neceſſity of introducing Fluxions of ſuperior orders. In his other excellent Works of this kind, which have been publiſh'd by himſelf, he makes expreſs mention of them, he diſcovers their nature and properties, and gives Rules for deriving their Equations. Therefore that this Work may be the more ſerviceable to Learners, and may fulfil the deſign of being an Inſtitution, I ſhall here make ſome inquiry into the nature of ſuperior Fluxions, and give ſome Rules for finding their Equations. And afterwards, in its proper place, I ſhall endeavour to ſhew ſomething of their application and uſe.

Now as the Fluxions of quantities which have been hitherto conſider'd, or their comparative Velocities of increaſe and decreaſe, are themſelves, and of their own nature, variable and flowing quantities alſo, and as ſuch are themſelves capable of perpetual increaſe and decreaſe, or of perpetual acceleration and retardation; they may be treated as other flowing quantities, and the relation of their Fluxions may be inquired and diſcover'd. In order to which we will adopt our Author's Notation already publiſh'd, in which we are to conceive, that as x, y, z, &c. have their Fluxions $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. ſo theſe likewiſe have their Fluxions $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, &c. which are the ſecond Fluxions of x, y, z, &c. And theſe again, being ſtill variable quantities, have their Fluxions denoted by $\dddot{x}$, $\dddot{y}$, $\dddot{z}$, &c. which are the third Fluxions of x, y, z, &c. And theſe again, being ſtill flowing quantities, have their Fluxions $\ddddot{x}$, $\ddddot{y}$, $\ddddot{z}$, &c. which are the fourth Fluxions of x, y, z, &c. And ſo we may proceed to ſuperior orders, as far as there ſhall be occaſion. Then, when any Equation is propoſed, conſiſting of variable quantities, as the relation of its Fluxions may be found by what has been taught before; ſo by repeating only the ſame operation, and conſidering the Fluxions as flowing Quantities, the relation

relation of the ſecond Fluxions may be found. And the like for all higher orders of Fluxions.

Thus if we have the Equation $y^2 - ax = 0$, in which are the two Fluents y and x, we ſhall have the firſt Fluxional Equation $2y\dot{y} - a\dot{x} = 0$. And here, as we have the three Fluents y, $\dot{y}$, and $\dot{x}$, if we take the Fluxions again, we ſhall have the ſecond Fluxional Equation $2y\ddot{y} + 2\dot{y}^2 - a\ddot{x} = 0$. And here, as there are four Fluents y, $\dot{y}$, $\ddot{y}$, and $\ddot{x}$, if we take the Fluxions again, we ſhall have the third Fluxional Equation $2y\dddot{y} + 2\dot{y}\ddot{y} + 4\dot{y}\ddot{y} - a\dddot{x} = 0$, or $2y\dddot{y} + 6\dot{y}\ddot{y} - a\dddot{x} = 0$. And here, as there are five Fluents y, $\dot{y}$, $\ddot{y}$, $\dddot{y}$, and $\dddot{x}$, if we take the Fluxions again, we ſhall have the fourth Fluxional Equation $2y\ddddot{y} + 2\dot{y}\dddot{y} + 6\dot{y}\dddot{y} + 6\ddot{y}^2 - a\ddddot{x} = 0$, or $2y\ddddot{y} + 8\dot{y}\dddot{y} + 6\ddot{y}^2 - a\ddddot{x} = 0$. And here, as there are ſix Fluents y, $\dot{y}$, $\ddot{y}$, $\dddot{y}$, $\ddddot{y}$, and $\ddddot{x}$, if we take the Fluxions again, we ſhall have $2y\overset{\cdot\cdot\cdot\cdot\cdot}{y} + 2\dot{y}\ddddot{y} + 8\dot{y}\ddddot{y} + 8\ddot{y}\dddot{y} + 12\ddot{y}\dddot{y} - a\overset{\cdot\cdot\cdot\cdot\cdot}{x} = 0$, or $2y\overset{\cdot\cdot\cdot\cdot\cdot}{y} + 10\dot{y}\ddddot{y} + 20\ddot{y}\dddot{y} - a\overset{\cdot\cdot\cdot\cdot\cdot}{x} = 0$, for the fifth Fluxional Equation. And ſo on to the ſixth, ſeventh, &c.

Now the Demonſtration of this will proceed much after the manner as our Author's Demonſtration of firſt Fluxions, and is indeed virtually included in it. For in the given Equation $y^2 - ax = 0$, if we ſuppoſe y and x to become at the ſame time $y + \dot{y}o$ and $x + \dot{x}o$, (that is, if we ſuppoſe $\dot{y}o$ and $\dot{x}o$ to denote the ſynchronal Moments of the Fluents y and x,) then by ſubſtitution we ſhall have $\overline{y + \dot{y}o}|^2 - a \times \overline{x + \dot{x}o} = 0$, or *in terminis expanſis*, $y^2 + 2y\dot{y}o + \dot{y}^2o^2 - ax - a\dot{x}o = 0$. Where expunging $y^2 - ax = 0$, and $\dot{y}^2o^2$, and dividing the reſt by o, it will be $2y\dot{y} - a\dot{x} = 0$ for the firſt fluxional Equation. Now in this Equation, if we ſuppoſe the ſynchronal Moments of the Fluents y, $\dot{y}$, and $\dot{x}$, to be $\dot{y}o$, $\ddot{y}o$, and $\ddot{x}o$ reſpectively; for thoſe Fluents we may ſubſtitute $\overline{y + \dot{y}o}$, $\overline{\dot{y} + \ddot{y}o}$, and $\overline{\dot{x} + \ddot{x}o}$ in the laſt Equation, and it will become $\overline{2y + 2\dot{y}o} \times \overline{\dot{y} + \ddot{y}o} - a \times \overline{\dot{x} + \ddot{x}o} = 0$, or expanding, $2y\dot{y} + 2\dot{y}\dot{y}o + 2y\ddot{y}o + 2\dot{y}\ddot{y}oo - a\dot{x} - a\ddot{x}o = 0$. Here becauſe $2y\dot{y} - a\dot{x} = 0$ by the given Equation, and becauſe $2\dot{y}\ddot{y}oo$ vaniſhes; divide the reſt by o, and we ſhall have $2\dot{y}^2 + 2y\ddot{y} - a\ddot{x} = 0$ for the ſecond fluxional Equation. Again in this Equation, if we ſuppoſe the Synchronal Moments of the Fluents y, $\dot{y}$, $\ddot{y}$, and $\ddot{x}$, to be $\dot{y}o$, $\ddot{y}o$, $\dddot{y}o$, and $\dddot{x}o$ reſpectively; for thoſe Fluents we may ſubſtitute $y + \dot{y}o$, $\dot{y} + \ddot{y}o$, $\ddot{y} + \dddot{y}o$, and $\ddot{x} + \dddot{x}o$ in the laſt Equation, and it will become $2 \times \overline{\dot{y} + \ddot{y}o}|^2 + \overline{2y + 2\dot{y}o} \times \overline{\ddot{y} + \dddot{y}o} - a \times \overline{\ddot{x} + \dddot{x}o} = 0$, or expanding and collecting, $2\dot{y}^2 + 6\dot{y}\ddot{y}o + 2\ddot{y}^2o^2 + 2y\ddot{y} + 2y\dddot{y}o + 2\dot{y}\dddot{y}o^2 - a\ddot{x} - a\dddot{x}o = 0$. But here becauſe $2\dot{y}^2 + 2y\ddot{y} - a\ddot{x} = 0$ by the laſt Equation; dividing the reſt by o, and expunging all the Terms in which o will ſtill be found, we ſhall have $6\dot{y}\ddot{y} + 2y\dddot{y} - a\dddot{x} = 0$ for the third fluxional Equation. And in like manner for all other orders of Fluxions, and for all other Examples. Q. E. D.

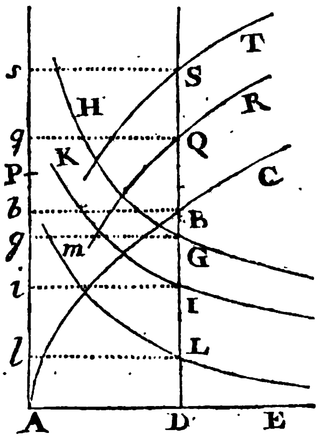
To illuſtrate the method of finding ſuperior Fluxions by another Example, let us take our Author's Equation $x^3 - ax^2 + axy - y^3 = 0$, in which he has found the ſimpleſt relation of the Fluxions to be $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + ax\dot{y} - 3\dot{y}y^2 = 0$. Here we have the flowing quantities x, y, $\dot{x}$, $\dot{y}$; and by the ſame Rules the Fluxion of this Equation, when contracted, will be $3\ddot{x}x^2 + 6\dot{x}^2x - 2a\ddot{x}x - 2a\dot{x}^2 + a\ddot{x}y + 2a\dot{x}\dot{y} + ax\ddot{y} - 3\ddot{y}y^2 - 6\dot{y}^2y = 0$. And in this Equation we have the flowing quantities x, y, $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$, ſo that taking the Fluxions again by the ſame Rules, we ſhall have the Equation, when contracted, $3\dddot{x}x^2 + 18\ddot{x}\dot{x}x + 6\dot{x}^3 - 2a\dddot{x}x - 6a\ddot{x}\dot{x} + a\dddot{x}y + 3a\ddot{x}\dot{y} + 3a\dot{x}\ddot{y} + ax\dddot{y} - 3\dddot{y}y^2 - 18\ddot{y}\dot{y}y - 6\dot{y}^3 = 0$. And as in this Equation there are found the flowing quantities x, y, $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$, $\dddot{x}$, $\dddot{y}$, we might proceed in like manner to find the relations of the fourth Fluxions belonging to this Equation, and all the following orders of Fluxions.

And here it may not be amiſs to obſerve, that as the propoſed Equation expreſſes the conſtant relation of the variable quantities x and y; and as the firſt fluxional Equation expreſſes the conſtant relation of the variable (but finite and aſſignable) quantities $\dot{x}$ and $\dot{y}$, which denote the comparative Velocity of increaſe or decreaſe of x and y in the propoſed Equation: So the ſecond fluxional Equation will expreſs the conſtant relation of the variable (but finite and aſſignable) quantities $\ddot{x}$ and $\ddot{y}$, which denote the comparative Velocity of the increaſe or decreaſe of $\dot{x}$ and $\dot{y}$ in the foregoing Equation. And in the third fluxional Equation we have the conſtant relation of the variable (but finite and aſſignable) quantities $\dddot{x}$ and $\dddot{y}$, which will denote the comparative Velocity of the increaſe or decreaſe of $\ddot{x}$ and $\ddot{y}$ in the foregoing Equation. And ſo on for ever. Here the Velocity of a Velocity, however uncouth it may ſound, will be no abſurd Idea when rightly conceived, but on the contrary will be a very rational and intelligible Notion. If there be ſuch a thing as Motion any how continually accelerated, that continual Acceleration will be the Velocity of a Velocity; and as that variation may be continually varied, that is, accelerated or retarded, there will be in nature, or at leaſt in the Underſtanding, the Velocity of a Velocity of a Velocity. Or in other words, the Notion of ſecond, third, and higher Fluxions, muſt be admitted as ſound and genuine: But to proceed:

We may much abbreviate the Equations now derived, by the known Laws of Analyticks. From the given Equation $x^3 - ax^2 + axy - y^3 = 0$ there is found a new Equation, wherein, becauſe of two new Symbols $\dot{x}$ and $\dot{y}$ introduced, we are at liberty to aſſume another Equation, beſides this now found, in order to a juſt Determination. For ſimplicity-ſake we may make $\dot{x}$ Unity, or any other conſtant quantity; that is, we may ſuppoſe x to flow equably, and therefore its Velocity is uniform. Make therefore $\dot{x} = 1$, and the firſt fluxional Equation will become $3x^2 - 2ax + ay + ax\dot{y} - 3\dot{y}y^2 = 0$. So in the Equation $3\ddot{x}x^2 + 6\dot{x}^2x - 2a\ddot{x}x - 2a\dot{x}^2 + a\ddot{x}y + 2a\dot{x}\dot{y} + ax\ddot{y} - 3\ddot{y}y^2 - 6\dot{y}^2y = 0$ there are four new Symbols introduced, $\dot{x}$, $\dot{y}$, $\ddot{x}$, and $\ddot{y}$, and therefore we may aſſume two other congruous Equations, which together with the two now found, will amount to a compleat Determination. Thus if for the ſake of ſimplicity we make one to be $\dot{x} = 1$, the other will neceſſarily be $\ddot{x} = 0$; and theſe being ſubſtituted, will reduce the ſecond fluxional Equation to this, $6x - 2a + 2a\dot{y} + ax\ddot{y} - 3\ddot{y}y^2 - 6\dot{y}^2y = 0$. And thus in the next Equation, wherein there are ſix new Symbols $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$, $\dddot{x}$, $\dddot{y}$; beſides the three Equations now found, we may take $\dot{x} = 1$, and thence $\ddot{x} = 0$, $\dddot{x} = 0$, which will reduce it to $6 + 3a\ddot{y} + ax\dddot{y} - 3\dddot{y}y^2 - 18\ddot{y}\dot{y}y - 6\dot{y}^3 = 0$. And the like of Equations of ſucceeding orders.

But all theſe Reductions and Abbreviations will be beſt made as the Equations are derived. Thus the propoſed Equation being $x^3 - ax^2 + axy - y^3 = 0$, taking the Fluxions, and at the ſame time making $\dot{x} = 1$, (and conſequently $\ddot{x}$, $\dddot{x}$, &c. $= 0$,) we ſhall have $3x^2 - 2ax + ay + ax\dot{y} - 3\dot{y}y^2 = 0$. And taking the Fluxions again, it will be $6x - 2a + 2a\dot{y} + ax\ddot{y} - 3\ddot{y}y^2 - 6\dot{y}^2y = 0$. And taking the Fluxions again, it will be $6 + 3a\ddot{y} + ax\dddot{y} - 3\dddot{y}y^2 - 18\ddot{y}\dot{y}y - 6\dot{y}^3 = 0$. And taking the Fluxions again, it will be $4a\dddot{y} + ax\ddddot{y} - 3\ddddot{y}y^2 - 24\dddot{y}\dot{y}y - 18\ddot{y}^2y - 36\ddot{y}\dot{y}^2 = 0$. And ſo on, as far as there is occaſion.

But now for the clearer apprehenſion of theſe ſeveral orders of Fluxions, I ſhall endeavour to illuſtrate them by a Geometrical Figure, adapted to a ſimple and a particular caſe. Let us aſſume the Equation $y^2 = ax$, or $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$, which will therefore belong to the Parabola ABC, whoſe Parameter is $AP = a$, Abſciſs $AD = x$, and Ordinate $BD = y$; where AP is a Tangent at the Vertex A. Then taking the Fluxions, we ſhall have $\dot{y} = \frac{1}{2}a^{\frac{1}{2}}\dot{x}x^{-\frac{1}{2}}$. And ſuppoſing the Parabola to be deſcribed by the equable motion of the Ordinate upon the Abſciſs, that equable Velocity may be expounded by the given Line or Parameter a, that is, we may put $\dot{x} = a$. Then it will be $\dot{y} = (\frac{1}{2}a^{\frac{3}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{3}{2}}}{2x^{\frac{1}{2}}} = \frac{a^{\frac{3}{2}}x^{\frac{1}{2}}}{2x} =)\ \frac{ay}{2x}$, which will give us this Conſtruction. Make x (AD) : y (BD) :: $\frac{1}{2}a$ ($\frac{1}{2}$AP) : $DG = \frac{ay}{2x} = \dot{y}$, and the Line DG will therefore repreſent the Fluxion of y or BD. And if this be done every where upon AE, (or if the Ordinate DG be ſuppos'd to move upon AE with a parallel motion,) a Curve GH will be conſtructed or deſcribed, whoſe Ordinates will every where expound the Fluxions of the correſponding Ordinates of the Parabola ABC. This Curve will be one of the Hyperbola's between the Aſymptotes AE and AP; for its Equation is $\dot{y} = \frac{a^{\frac{3}{2}}}{2x^{\frac{1}{2}}}$, or $\dot{y}\dot{y} = \frac{a^3}{4x}$.



Again, from the Equation $\dot{y} = \frac{ay}{2x}$, or $2x\dot{y} = ay$, by taking the Fluxions again, and putting $\dot{x} = a$ as before, we ſhall have $2a\ddot{y} + 2x\ddot{y} = a\dot{y}$, or $-\ddot{y} = \frac{a\dot{y}}{2x}$; where the negative ſign ſhews only, that $\ddot{y}$ is to be conſider'd rather as a retardation than an acceleration, or an acceleration the contrary way. Now this will give us the following Conſtruction. Make x (AD) : $\dot{y}$ (DG) :: $\frac{1}{2}a$ ($\frac{1}{2}$AP) : DI $= \ddot{y}$, and the Line DI will therefore repreſent the Fluxion of DG, or of $\dot{y}$, and therefore the ſecond Fluxion of BD, or of y. And if this be done every where upon AE, a Curve IK will be conſtructed, whoſe Ordinates will always expound the ſecond Fluxions of the correſponding Ordinates of the Parabola ABC. This Curve likewiſe will be one of the Hyperbola's, for its Equation is $-\ddot{y} = \frac{a\dot{y}}{2x} = \frac{a^{\frac{5}{2}}}{4x^{\frac{3}{2}}}$, or $\ddot{y}\ddot{y} = \frac{a^5}{16x^3}$.

Again, from the Equation $-\ddot{y} = \frac{a\dot{y}}{2x}$, or $-2x\ddot{y} = a\dot{y}$, by taking the Fluxions we ſhall have $-2a\ddot{y} - 2x\dddot{y} = a\ddot{y}$, or $-\dddot{y} = \frac{3a\ddot{y}}{2x}$, which will give us this Conſtruction. Make x (AD) : $\ddot{y}$ (DI) :: $\frac{3}{2}a$ ($\frac{3}{2}$AP) : DL $= \dddot{y}$, and the Line DL will therefore repreſent the Fluxion of DI, or of $\ddot{y}$, the ſecond Fluxion of DG, or of $\dot{y}$, and the third Fluxion of BD, or of y. And if this be done every where upon AE, a Curve LM will be conſtructed, whoſe Ordinates will always expound the third Fluxions of the correſponding Ordinates of the Parabola ABC. This Curve will be an Hyperbola, and its Equation will be $-\dddot{y} = \frac{3a\ddot{y}}{2x} = \frac{3a^{\frac{7}{2}}}{8x^{\frac{5}{2}}}$, or $\dddot{y}\dddot{y} = \frac{9a^7}{64x^5}$. And ſo we might proceed to conſtruct Curves, the Ordinates of which (in the preſent Example) would expound or repreſent the fourth, fifth, and other orders of Fluxions.

We might likewiſe proceed in a retrograde order, to find the Curves whoſe Ordinates ſhall repreſent the Fluents of any of theſe Fluxions, when given. As if we had $\dot{y} = \frac{a^{\frac{1}{2}}}{2x^{\frac{1}{2}}} = \frac{1}{2}a^{\frac{1}{2}}\dot{x}x^{-\frac{1}{2}}$, or if the Curve GH were given; by taking the Fluents, (as will be taught in the next Problem,) it would be $y = (a^{\frac{1}{2}}x^{\frac{1}{2}} = \frac{a^{\frac{1}{2}}x}{x^{\frac{1}{2}}} =) \frac{2x\dot{y}}{a}$, which will give us this Conſtruction. Make $\frac{1}{2}a$ ($\frac{1}{2}$AP) : x (AD) :: $\dot{y}$ (DG) : DB $= \frac{2x\dot{y}}{a}$, and the Line DB will repreſent the Fluent of DG, or of $\dot{y}$. And if this be done every where upon the Line AE, a Curve AB will be conſtructed, whoſe Ordinates will always expound the Fluents of the correſponding Ordinates of the Curve GH. This Curve will be the common Parabola, whoſe

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Parameter is the Line AP $= a$. For its Equation is $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$, or $yy = ax$.

So if we had the Parabola ABC, we might conceive its Ordinates to repreſent Fluxions, of which the correſponding Ordinates DQ of ſome other Curve, ſuppoſe QR, would repreſent the Fluents. To find which Curve, put $\overset{\prime}{y}$ for the Fluent of y, $\overset{\prime\prime}{y}$ for the Fluent of $\overset{\prime}{y}$, &c. (That is, let, &c. $\overset{\prime\prime\prime}{y}$, $\overset{\prime\prime}{y}$, $\overset{\prime}{y}$, y, $\dot{y}$, $\ddot{y}$, $\dddot{y}$, &c. be a Series of Terms proceeding both ways indefinitely, of which every ſucceeding Term repreſents the Fluxion of the preceding, and *vice verſâ*; according to a Notation of our Author's, deliver'd elſewhere.) Then becauſe it is $y = (a^{\frac{1}{2}}x^{\frac{1}{2}} = a^{\frac{1}{2}}x^{\frac{1}{2}} \times \frac{\dot{x}}{a} =) \frac{\dot{x}x^{\frac{1}{2}}}{a^{\frac{1}{2}}}$, taking the Fluents it will be $\overset{\prime}{y} = \left(\frac{2x^{\frac{3}{2}}}{3a^{\frac{1}{2}}} = \frac{2a^{\frac{1}{2}}x^{\frac{1}{2}}}{3a} =\right) \frac{2xy}{3a}$; which will give us this Conſtruction. Make $\frac{3}{2}a$ ($\frac{3}{2}$AP) : x (AD) :: y (BD) : $\frac{2xy}{3a} = \overset{\prime}{y} =$ DQ, and the Line DQ will repreſent the Fluent of DB, or of y. And if the ſame be done at every point of the Line AE, a Curve QR will be form'd, the Ordinates of which will always expound the Fluents of the correſponding Ordinates of the Parabola ABC. This Curve alſo will be a Parabola, but of a higher order, the Equation of which is $\overset{\prime}{y} = \frac{2x^{\frac{3}{2}}}{3a^{\frac{1}{2}}}$, or $\overset{\prime}{y}\overset{\prime}{y} = \frac{4x^3}{9a}$.

Again, becauſe $\overset{\prime}{y} = \left(\frac{2x^{\frac{3}{2}}}{3a^{\frac{1}{2}}} = \frac{2x^{\frac{3}{2}} \times \dot{x}}{3a^{\frac{1}{2}} \times a} =\right) \frac{2\dot{x}x^{\frac{3}{2}}}{3a^{\frac{3}{2}}}$; taking the Fluents it will be $\overset{\prime\prime}{y} = \left(\frac{4x^{\frac{5}{2}}}{15a^{\frac{3}{2}}} = \frac{2x^{\frac{3}{2}}}{3a^{\frac{1}{2}}} \times \frac{2x}{5a} =\right) \frac{2x\overset{\prime}{y}}{5a}$, which will give us this Conſtruction. Make $\frac{5}{2}a$ ($\frac{5}{2}$AP) : x (AD) :: $\overset{\prime}{y}$ (DQ) : $\frac{2x\overset{\prime}{y}}{5a} = \overset{\prime\prime}{y} =$ DS, and the Line DS will repreſent the Fluent of DQ, or of $\overset{\prime}{y}$. And if the ſame be done at every point of the Line AE, a Curve ST will thereby be form'd, the Ordinates of which will expound the Fluents of the correſponding Ordinates of the Curve QR. This Curve will be a Parabola, whoſe Equation is $\overset{\prime\prime}{y} = \frac{4x^{\frac{5}{2}}}{15a^{\frac{3}{2}}}$, or $\overset{\prime\prime}{y}\overset{\prime\prime}{y} = \frac{16x^5}{225a^3}$. And ſo we might go on as far as we pleaſe.

Laſtly, if we conceive DB, the common Ordinate of all theſe Curves, to be any where thus conſtructed upon AD, that is, to be thus divided in the points S, Q, B, G, I, L, &c. from whence to AP are drawn S*s*, Q*q*, B*b*, G*g*, I*i*, L*l*, &c. parallel to AE; and if this Ordinate be farther conceived to move either backwards or forwards upon AE, with an equable Velocity, (repreſented by $AP = a = \dot{x}$,) and as it deſcribes theſe Curves, to carry the aforeſaid Parallels along with it in its motion: Then the points *s*, *q*, *b*, *g*, *i*, *l*, &c. will likewiſe move in ſuch a manner, in the Line AP, as that the Velocity of each point will be repreſented by the diſtance of the next from the point A. Thus the Velocity of *s* will be repreſented by A*q*, the Velocity of *q* by A*b*, of *b* by A*g*, of *g* by A*i*, of *i* by A*l*, &c. Or in other words, A*q* will be the Fluxion of A*s*; A*b* will be the Fluxion of A*q*, or the ſecond Fluxion of A*s*; A*g* will be the Fluxion of A*b*, or the ſecond Fluxion of A*q*, or the third Fluxion of A*s*; A*i* will be the Fluxion of A*g*, or the ſecond Fluxion of A*b*, or the third Fluxion of A*q*, or the fourth Fluxion of A*s*; and ſo on. Now in this inſtance the ſeveral orders of Fluxions, or Velocities, are not only expounded by their Proxies and Repreſentatives, but alſo are themſelves actually exhibited, as far as may be done by Geometrical Figures. And the like obtains wherever elſe we make a beginning; which ſufficiently ſhews the relative nature of all theſe orders of Fluxions and Fluents, and that they differ from each other by mere relation only, and in the manner of conceiving. And in general, what has been obſerved from this Example, may be eaſily accommodated to any other caſes whatſoever.

Or theſe different orders of Fluents and Fluxions may be thus explain'd abſtractedly and Analytically, without the aſſiſtance of Curve-lines, by the following general Example. Let any conſtant and known quantity be denoted by a, and let a^m be any given Power or Root of the ſame. And let x^m be the like Power or Root of the variable and indefinite quantity x. Make $a^m : x^m :: a : y$, or $y = \frac{ax^m}{a^m} = a^{1-m}x^m$. Here y alſo will be an indefinite quantity, which will become known as ſoon as the value of x is aſſign'd. Then taking the Fluxions, it will be $\dot{y} = ma^{1-m}\dot{x}x^{m-1}$; and ſuppoſing x to flow or increaſe uniformly, and making its conſtant Velocity or Fluxion $\dot{x} = a$, it will be $\dot{y} = ma^{2-m}x^{m-1}$. Here if for $a^{1-m}x^m$ we write its value y, it will be $\dot{y} = \frac{may}{x}$, that is, $x : ma :: y : \dot{y}$. So that $\dot{y}$ will be alſo a known and aſſignable Quantity,

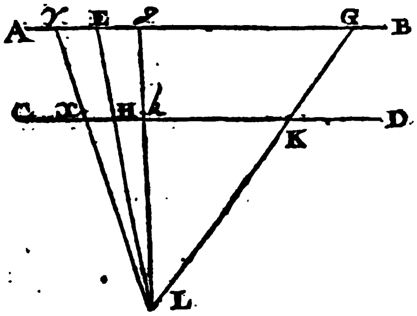
tity, whenever x (and therefore y) is aſſign'd. Then taking the Fluxions again, we ſhall have $\ddot{y} = m \times \overline{m-1}\,a^{2-m}\dot{x}x^{m-2} = m \times \overline{m-1}\,a^{3-m}x^{m-2}$; or for $ma^{2-m}x^{m-1}$ writing its value $\dot{y}$, it will be $\ddot{y} = \frac{\overline{m-1}\,a\dot{y}}{x}$, that is, $x : \overline{m-1}\,a :: \dot{y} : \ddot{y}$. So that $\ddot{y}$ will become a known quantity, when x (and therefore y and $\dot{y}$) is aſſign'd. Then taking the Fluxions again, we ſhall have $\dddot{y} = m \times \overline{m-1} \times \overline{m-2}\,a^{4-m}x^{m-3}$, or $\dddot{y} = \frac{\overline{m-2}\,a\ddot{y}}{x}$, that is, $x : \overline{m-2}\,a :: \ddot{y} : \dddot{y}$; where alſo $\dddot{y}$ will be known, when x is given. And taking the Fluxions again, we ſhall have $\ddddot{y} = m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3}\,a^{5-m}x^{m-4} = \frac{\overline{m-3}\,a\dddot{y}}{x}$; that is, $x : \overline{m-3}\,a :: \dddot{y} : \ddddot{y}$. So that $\ddddot{y}$ will alſo be known, whenever x is given. And from this Induction we may conclude in general, that if the order of Fluxions be denoted by any integer number n, or if n be put for the number of points over the Letter y, it will always be $x : \overline{m-n}\,a :: \overset{n}{y} : \overset{n+1}{y}$; or from the Fluxion of any order being given, the Fluxion of the next immediate order may be hence found.

Or we may thus invert the proportion $\overline{m-n}\,a : x :: \overset{n+1}{y} : \overset{n}{y}$, and then from the Fluxion given, we ſhall find its next immediate Fluent. As if $n = 2$, 'tis $\overline{m-2}\,a : x :: \dddot{y} : \ddot{y}$. If $n = 1$, 'tis $\overline{m-1}\,a : x :: \ddot{y} : \dot{y}$. If $n = 0$, 'tis $ma : x :: \dot{y} : y$. And obſerving the ſame analogy, if $n = -1$, 'tis $\overline{m+1}\,a : x :: y : \overset{\prime}{y}$; where $\overset{\prime}{y}$ is put for the Fluent of y, or for y with a negative point. And here becauſe $y = a^{1-m}x^{m}$, it will be $\overline{m+1}\,a : x :: a^{1-m}x^{m} : \overset{\prime}{y}$, or $\overset{\prime}{y} = \frac{a^{1-m}x^{m+1}}{\overline{m+1}\,a} = \frac{x^{m+1}}{\overline{m+1}\,a^{m}}$: which alſo may thus appear. Becauſe $y = (a^{1-m}x^{m} = \frac{a^{1-m}\dot{x}x^{m}}{a} =) \frac{\dot{x}x^{m}}{a^{m}}$, taking the Fluents, (ſee the next Problem,) it will be $\overset{\prime}{y} = \frac{x^{m+1}}{\overline{m+1}\,a^{m}}$. Again, if we make $n = -2$, 'tis $\overline{m+2}\,a : x :: \overset{\prime}{y} : \overset{\prime\prime}{y}$, or $\overset{\prime\prime}{y} = \frac{x\overset{\prime}{y}}{\overline{m+2}\,a} = \frac{x^{m+2}}{\overline{m+1} \times \overline{m+2}\,a^{m+1}}$. For becauſe $\overset{\prime}{y} = \frac{x^{m+1}}{\overline{m+1}\,a^{m}} \times \frac{\dot{x}}{a} = \frac{\dot{x}x^{m+1}}{\overline{m+1}\,a^{m+1}}$, taking the Fluents it will be $\overset{\prime\prime}{y} = \frac{x^{m+2}}{\overline{m+1} \times \overline{m+2}\,a^{m+1}}$. Again, if we make $n = -3$, 'tis $\overline{m+3}a : x :: \overset{\prime\prime}{y} : \overset{\prime\prime\prime}{y}$, or $\overset{\prime\prime\prime}{y} = \frac{x\overset{\prime\prime}{y}}{\overline{m+3}a} = \frac{x^{m+3}}{\overline{m+1} \times \overline{m+2} \times \overline{m+3}\,a^{m+2}}$. And ſo for all other ſuperior orders of Fluents.

And this may ſuffice in general, to ſhew the comparative nature and properties of theſe ſeveral orders of Fluxions and Fluents, and to teach the operations by which they are produced, or to find their reſpective fluxional Equations. As to the uſes they may be apply'd to, when found, that will come more properly to be conſider'd in another place.

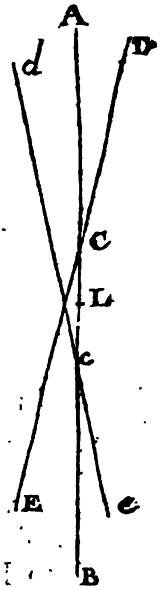
SECT. III. *The Geometrical and Mechanical Elements of Fluxions.*

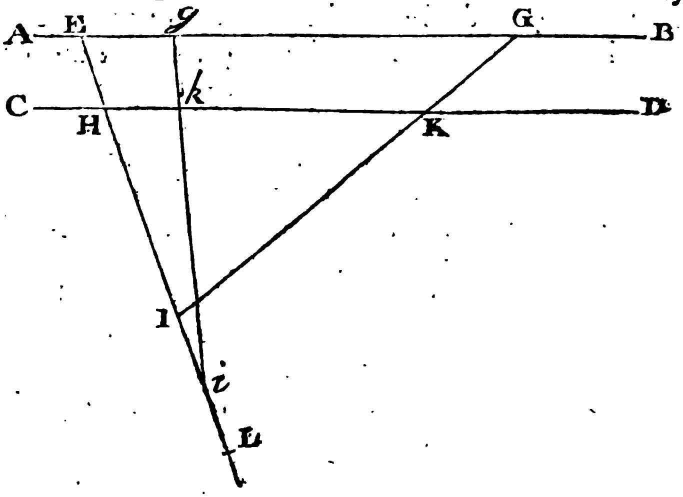
THE foregoing Principles of the Doctrine of Fluxions being chiefly abſtracted and Analytical, I ſhall here endeavour, after a general manner, to ſhew ſomething analogous to them in Geometry and Mechanicks; by which they may become, not only the object of the Underſtanding, and of the Imagination, (which will only prove their poſſible exiſtence,) but even of Senſe too, by making them actually to exiſt in a viſible and ſenſible form. For it is now become neceſſary to exhibit them all manner of ways, in order to give a ſatisfactory proof, that they have indeed any real exiſtence at all.



And firſt, by way of preparation, it will be convenient to conſider uniform and equable motions, as alſo ſuch as are alike inequable. Let the right Line AB be deſcribed by the equable motion of a point, which is now at E, and will preſently be at G. Alſo let the Line CD, parallel to the former, be deſcribed by the equable motion of a point, which is in H and K, at the ſame times as the former is in E and G. Then will EG and HK be contemporaneous Lines, and therefore will be proportional to the Velocity of each moving point reſpectively. Draw the indefinite Lines EH and GK, meeting in L; then becauſe of like Triangles ELG and HLK, the Velocities of the points E and H, which were before as EG and HK, will be now as EL and HL. Let the deſcribing points G and K be conceived to move back again, with the ſame Velocities, towards A and C, and before they approach to E and H let them be found in g and k, at any ſmall diſtance from E and H, and draw gk, which will paſs through L; then ſtill their Velocities will be in the ratio of Eg and Hk, be thoſe Lines ever ſo little, that is, in the ratio of EL and HL. Let the moving points g and k continue to move till they coincide with E and H; in which caſe the decreaſing Lines Eg and Hk will paſs through all poſſible magnitudes that are leſs and leſs, and will finally become vaniſhing Lines. For they muſt intirely vaniſh at the ſame moment, when the points g and k ſhall coincide with E and H. In all which ſtates and circumſtances they will ſtill retain the ratio of EL to HL, with which at laſt they will finally vaniſh. Let thoſe points ſtill continue to move, after they have coincided with E and H, and let them be found again at the ſame time in γ and $\varkappa$, at any diſtance beyond E and H. Still the Velocities, which are now as $E\gamma$ and $H\varkappa$, and may be eſteemed negative, will be as EL and HL, whether thoſe Lines $E\gamma$ and $H\varkappa$ are of any finite magnitude, or are only naſcent Lines; that is, if the Line $\gamma\varkappa L$, by its angular motion, be but juſt beginning to emerge and divaricate from EHL. And thus it will be when both theſe motions are equable motions, as alſo when they are alike inequable; in both which caſes the common interſection of all the Lines EHL, GKL, gkL, &c. will be the fixt point L. But when either or both theſe motions are ſuppos'd to be inequable motions, or to be any how continually accelerated or retarded, theſe Symptoms will be ſomething different; for then the point L, which will ſtill be the common interſection of thoſe Lines when they firſt begin to coincide, or to divaricate, will no longer be a fixt but a moveable point, and an account muſt be had of its motion. For this purpoſe we may have recourſe to the following Lemma.

Let AB be an indefinite and fixt right Line, along which another indefinite but moveable right Line DE may be conceived to move or roll in ſuch a manner, as to have both a progreſſive motion, as alſo an angular motion about a moveable Center C. That is, the common interſection C of the two Lines AB and DE may be ſuppoſed to move with any progreſſive motion from A towards B, while at the ſame time the moveable Line DE revolves about the ſame point C, with any angular motion. Then as the Angle ACD continually decreaſes, and at laſt vaniſhes when the two Lines ACB and DCE coincide; yet even then the point of interſection C, (as it may be ſtill call'd,) will not be loſt and annihilated, but will appear again, as ſoon as the Lines begin to divaricate, or to ſeparate from each other. That is, if C be the point of interſection before the coincidence, and *c* the point of interſection after the coincidence, when the Line *dce* ſhall again emerge out of AB; there will be ſome intermediate point L, in which C and *c* were united in the ſame point, at the moment of coincidence. This point, for diſtinction-ſake, may be call'd the *Node*, or *the point of no divarication*. Now to apply this to inequable Motions;

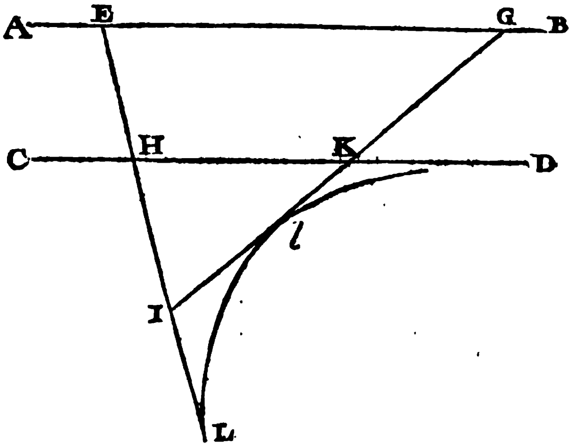




Let the Line AB be deſcribed by the continually accelerated motion of a point, which is now in E, and will be preſently found in G. Alſo let the Line CD, parallel to the former, be deſcribed by the equable motion of a point, which is found in H and K, at the ſame times as the other point is in E and G. Then will EG and HK be contemporaneous Lines; and producing EH and GK till they meet in I, thoſe contemporaneous Lines will be as EI and HI reſpectively. Let the deſcribing points G and K be conceived to move back again towards A and C, each with the ſame degrees of Velocity, in every point of their motion, as they had before acquired; and let them arrive at the ſame time at *g* and *k*, at ſome ſmall diſtance from E and H, and draw *gki* meeting EH in *i*. Then E*g* and H*k*, being contemporary Lines alſo, and very little by ſuppoſition, they will be nearly as the Velocities at g and k, that is, at E and H; which contemporary Lines will be now as Ei and Hi. Let the points g and k continue their motion till they coincide with E and H, or let the Line GKI or gki continue its progreſſive and angular motion in this manner, till it coincides with EHL, and let L be the Node, or point of no divarication, as in the foregoing Lemma. Then will the laſt ratio of the vaniſhing Lines Eg and Hk, which is the ratio of the Velocities at E and H, be as EL and HL reſpectively.

Hence we have this Corollary. If the point E (in the foregoing figure,) be ſuppos'd to move from A towards B, with a Velocity any how accelerated, and at the ſame time the point H moves from C towards D with an equable Velocity, (or inequable, if you pleaſe;) thoſe Velocities in E and H will be reſpectively as the Lines EL and HL, which point L is to be found, by ſuppoſing the contemporary Lines EG and HK continually to diminiſh, and finally to vaniſh. Or by ſuppoſing the moveable indefinite Line GKI to move with a progreſſive and angular motion, in ſuch manner, as that EG and HK ſhall always be contemporary Lines, till at laſt GKI ſhall coincide with the Line EHL, at which time it will determine the Node L, or the point of no divarication. So that if the Lines AE and CH repreſent two Fluents, any how related, their Velocities of deſcription at E and H, or their reſpective Fluxions, will be in the ratio of EL and HL.

And hence it will follow alſo, that the Locus of the moveable point or Node L, that is, of all the points of no divarication, will be ſome Curve-line Ll, to which the Lines EHL and GKl will always be Tangents in L and l. And the nature of this Curve Ll may be determined by the given relation of the Fluents or Lines AE and CH; and *vice verſâ*. Or however the relation of its intercepted Tangents EL and HL may be determined in all caſes; that is, the ratio of the Fluxions of the given Fluents.



For illuſtration-ſake, let us apply this to an Example. Make the Fluents $AE = y$ and $CH = x$, and let the relation of theſe be always expreſs'd by this Equation $y = x^n$. Make the contemporary Lines $EG = Y$ and $HK = X$; and becauſe AE and CH are contemporary by ſuppoſition, we ſhall have the whole Lines AG and CK contemporary alſo, and thence the Equation $y + Y = \overline{x + X}|^n$. This by our Author's Binomial Theorem will produce $y + Y = x^n + nx^{n-1}X + n \times \frac{n-1}{2}x^{n-2}X^2$, &c. which (becauſe $y = x^n$) will become $Y = nx^{n-1}X + n \times \frac{n-1}{2}x^{n-2}X^2$, &c. or in an Analogy, $X : Y :: 1 : nx^{n-1} + n \times \frac{n-1}{2}x^{n-2}X$, &c. which will be the general relation of the contemporary Lines or Increments EG and HK. Now let us ſuppoſe the indefinite Line GKI, which limits theſe contemporary Lines, to return back by a progreſſive and angular motion, ſo as always to intercept contemporary Lines EG and HK, and finally to coincide with EHL, and by that means to determine the Node L; that is, we may ſuppoſe $EG = Y$ and $HK = X$, to diminiſh *in infinitum*, and to become vaniſhing Lines, in which caſe we ſhall have $X : Y :: 1 : nx^{n-1}$. But then it will be likewiſe $X : Y :: HK : EG :: HL : EL :: \dot{x} : \dot{y}$, or $1 : nx^{n-1} :: \dot{x} : \dot{y}$, or $\dot{y} = n\dot{x}x^{n-1}$.

And hence we may have an expedient for exhibiting Fluxions and Fluents Geometrically and Mechanically, in all circumſtances, ſo as to make them the objects of Senſe and ocular Demonſtration. Thus in the laſt figure, let the two parallel lines AB and CD be deſcribed by the motion of two points E and H, of which E moves any how inequably, and (if you pleaſe) H may be ſuppos'd to move equably and uniformly; and let the points H and K correſpond to E and G. Alſo let the relation of the Fluents $AE = y$ and $CH = x$ be defined by any Equation whatever. Suppoſe now the deſcribing points E and H to carry along with them the indefinite Line EHL, in all their motion, by which means the point or Node L will deſcribe ſome Curve L*l*, to which EL will always be a Tangent in L. Or ſuppoſe EHL to be the Edge of a Ruler, of an indefinite length, which moves with a progreſſive and angular motion thus combined together; the moveable point or Node L in this Line, which will have the leaſt angular motion, and which is always the point of no divarication, will deſcribe the Curve, and the Line or Edge itſelf will be a Tangent to it in L. Then will the ſegments EL and HL be proportional to the Velocity of the points E and H reſpectively; or will exhibit the ratio of the Fluxions $\dot{y}$ and $\dot{x}$, belonging to the Fluents $AE = y$ and $CF = x$.

Or if we ſuppoſe the Curve L*l* to be given, or already conſtructed, we may conceive the indefinite Line EHIL to revolve or roll about it, and by continually applying itſelf to it, as a Tangent, to move from the ſituation EHIL to GK*l*I. Then will AE and CH be the Fluents, the ſenſible velocities of the deſcribing points E and H will be their Fluxions, and the intercepted Tangents EL and HL will be the rectilinear meaſures of thoſe Fluxions or Velocities. Or it may be repreſented thus: If L*l* be any rigid obſtacle in form of a Curve, about which a flexible Line, or Thread, is conceived to be wound, part of which is ſtretch'd out into a right Line LE, which will therefore touch the Curve in L; if the Thread be conceived to be farther wound about the Curve, till it comes into the ſituation L*l*KG; by this motion it will exhibit, even to the Eye, the ſame increaſing Fluents as before, their Velocities of increaſe, or their Fluxions, as alſo the Tangents or rectilinear repreſentatives of thoſe Fluxions. And the ſame may be done by unwinding the Thread, in the manner of an Evolute. Or inſtead of the Thread we may make uſe of a Ruler, by applying its Edge continually to the curved Obſtacle L*l*, and making it any how revolve about the moveable point of Contact L or *l*. In all which manners the Fluents, Fluxions, and their rectilinear meaſures, will be ſenſibly and mechanically exhibited, and therefore they muſt be allowed to have a place *in rerum naturâ.* And if they are in nature, even tho' they were but barely poſſible and conceiveable, much more if they are ſenſible and viſible, it is the province of the Mathematicks, by ſome method or other, to inveſtigate and determine their properties and proportions.

Or as by one Thread EHL, perpetually winding about the curved obſtacle L*l*, of a due figure, we ſhall ſee the Fluents AE and CH continually to increaſe or decreaſe, at any rate aſſign'd, by the motion of the Thread EHL either backwards or forwards; and as we ſhall thereby ſee the comparative Velocities of the points E and H, that is, the Fluxions of the Fluents AE and CH, and alſo the Lines EL and HL, whoſe variable ratio is always the rectilinear meaſure of thoſe Fluxions: So by the help of another Thread GK*l*L, winding about the obſtacle in its part *l*L, and then ſtretching out into a right Line or Tangent *l*KG, and made to move backwards or forwards, as before; if the firſt Thread be at reſt in any given ſituation EHL, we may ſee the ſecond Thread deſcribe the contemporary Lines or Increments EG and HK, by which the Fluents AE and CH are continually increaſed; and if GK*l* is made to approach towards EHL, we may ſee thoſe contemporary Lines continually to diminiſh, and their ratio continually approaching towards the ratio of EL to HL; and continuing the motion, we may preſently ſee thoſe two Lines actually to coincide, or to unite as one Line, and then we may ſee the contemporary Lines actually to vaniſh at the ſame time, and their ultimate ratio actually to become that of EL to HL. And if the motion be ſtill continued, we ſhall ſee the Line GK*l* to emerge again out of EHL, and begin to deſcribe other contemporary Lines, whoſe naſcent proportion will be that of EL to HL. And ſo we may go on till the Fluents are exhauſted. All theſe particulars may be thus eaſily made the objects of ſight, or of Ocular Demonſtration.

This may ſtill be added, that as we have here exhibited and repreſented firſt Fluxions geometrically and mechanically, we may do the ſame thing, *mutatis mutandis*, by any higher orders of Fluxions. Thus if we conceive a ſecond figure, in which the Fluential Lines ſhall increaſe after the rate of the ratio of the intercepted Tangents (or the Fluxions) of the firſt figure; then its intercepted Tangents will expound the ratio of the ſecond Fluxions of the Fluents in the firſt figure. Alſo if we conceive a third figure, in which the Fluential Lines ſhall increaſe after the rate of the intercepted Tangents of the ſecond figure; then its intercepted Tangents will expound the third Fluxions of the Fluents in the firſt figure. And ſo on as far as we pleaſe. This is a neceſſary conſequence from the relative nature of theſe ſeveral orders of Fluxions, which has been ſhewn before.

And farther to ſhew the univerſality of this Speculation, and how well it is accommodated to explain and repreſent all the circumſtances of Fluxions and Fluents; we may here take notice, that it may be alſo adapted to thoſe caſes, in which there are more than two Fluents, which have a mutual relation to each other, expreſs'd by one or more Equations. For we need but introduce a third parallel Line, and ſuppoſe it to be deſcribed by a third point any how moving, and that any two of theſe deſcribing points carry an indefinite Line along with them, which by revolving as a Tangent, deſcribes the Curve whoſe Tangents every where determine the Fluxions. As alſo that any other two of thoſe three points are connected by another indefinite Line, which by revolving in like manner deſcribes another ſuch Curve. And ſo there may be four or more parallel Lines. All but one of theſe Curves may be aſſumed at pleaſure, when they are not given by the ſtate of the Queſtion. Or Analytically,

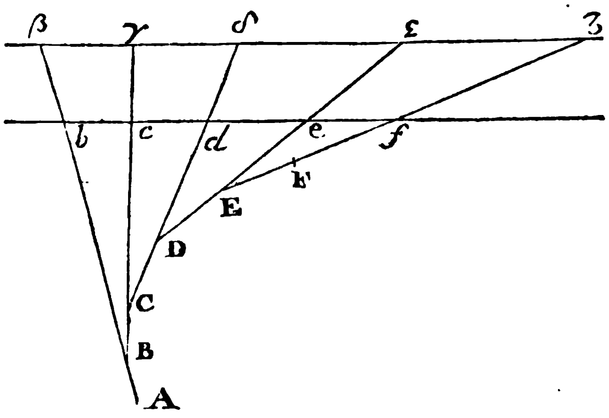
tically, ſo many Equations may be aſſumed, except one, (if not given by the Problem,) as is the number of the Fluents concern'd.

But laſtly, I believe it may not be difficult to give a pretty good notion of Fluents and Fluxions, even to ſuch Perſons as are not much verſed in Mathematical Speculations, if they are willing to be inform'd, and have but a tolerable readineſs of apprehenſion. This I ſhall here attempt to perform, in a familiar way, by the inſtance of a Fowler, who is aiming to ſhoot two Birds at once, as is repreſented in the Frontiſpiece. Let us ſuppoſe the right Line AB to be parallel to the Horizon, or level with the Ground, in which a Bird is now flying at G, which was lately at F, and a little before at E. And let this Bird be conceived to fly, not with an equable or uniform ſwiftneſs, but with a ſwiftneſs that always increaſes, (or with a Velocity that is continually accelerated,) according to ſome known rate. Let there alſo be another right Line CD, parallel to the former, at the ſame or any other convenient diſtance from the Ground, in which another Bird is now flying at K, which was lately at I, and a little before at H; juſt at the ſame points of time as the firſt Bird was at G, F, E, reſpectively. But to fix our Ideas, and to make our Conceptions the more ſimple and eaſy, let us imagine this ſecond Bird to fly equably, or always to deſcribe equal parts of the Line CD in equal times. Then may the equable Velocity of this Bird be uſed as a known meaſure, or ſtandard, to which we may always compare the inequable Velocity of the firſt Bird. Let us now ſuppoſe the right Line EH to be drawn, and continued to the point L, ſo that the proportion (or ratio) of the two Lines EL and HL may be the ſame as that of the Velocities of the two Birds, when they were at E and H reſpectively. And let us farther ſuppoſe, that the Eye of a Fowler was at the ſame time at the point L, and that he directed his Gun, or Fowling-piece, according to the right Line LHE, in hopes to ſhoot both the Birds at once. But not thinking himſelf then to be ſufficiently near, he forbears to diſcharge his Piece, but ſtill pointing it at the two Birds, he continually advances towards them according to the direction of his Piece, till his Eye is preſently at M, and the Birds at the ſame time in F and I, in the ſame right Line FIM. And not being yet near enough, we may ſuppoſe him to advance farther in the ſame manner, his Piece being always directed or level'd at the two Birds, while he himſelf walks forward according to the direction of his Piece, till his Eye is now at N, and the Birds in the ſame right Line with his Eye, at K and G. The Path of his Eye, deſcribed by this double motion, (or compounded of a progreſſive and angular motion,) will be ſome Curve-line LMN, in the ſame Plain as the reſt of the figure, which will have this property, that the proportion of the diſtances of his Eye from each Bird, will be the ſame every where as that of their reſpective Velocities. That is, when his Eye was at L, and the Birds at E and H, their Velocities were then as EL and HL, by the Conſtruction. And when his Eye was at M, and the Birds at F and I, their Velocities were in the ſame proportion as the Lines FM and IM, by the nature of the Curve LMN. And when his Eye is at N, and the Birds at G and K, their Velocities are in the proportion of GN to KN, by the nature of the ſame Curve. And ſo univerſally, of all other ſituations. So that the Ratio of thoſe two Lines will always be the ſenſible meaſure of the ratio of thoſe two ſenſible Velocities. Now if theſe Velocities, or the ſwiftneſſes of the flight of the two Birds in this inſtance, are call'd *Fluxions*; then the Lines deſcribed by the Birds in the ſame time, may be call'd their *contemporaneous Fluents*; and all inſtances whatever of Fluents and Fluxions, may be reduced to this Example, and may be illuſtrated by it.

And thus I would endeavour to give ſome notion of Fluents and Fluxions, to Perſons not much converſant in the Mathematicks; but ſuch as had acquired ſome ſkill in theſe Sciences, I would thus proceed farther to inſtruct, and to apply what has been now deliver'd. The contemporaneous Fluents being EF $= y$, and HI $= x$, and their rate of flowing or increaſing being ſuppos'd to be given or known; their relation may always be expreſs'd by an Equation, which will be compos'd of the variable quantities x and y, together with any known quantities. And that Equation will have this property, becauſe of thoſe variable quantities, that as FG and IK, EG and HK, and infinite others, are alſo contemporaneous Fluents; it will indifferently exhibit the relation of thoſe Lines alſo, as well as of EF and HI; or they may be ſubſtituted in the Equation, inſtead of x and y. And hence we may derive a Method for determining the Velocities themſelves, or for finding Lines proportional to them. For making FG $=$ Y, and IK $=$ X; in the given Equation I may ſubſtitute $y + $ Y inſtead of y, and $x + $ X inſtead of x, by which I ſhall obtain an Equation, which in all circumſtances will exhibit the relation of thoſe Quantities or Increments. Now it may be plainly perceived, that if the Line MIF is conceived continually to approach nearer and nearer to the Line NKG, (as juſt now, in the inſtance of the Fowler,) till it finally coincides with it; the Lines FG $=$ Y, and IK $=$ X, will continually decreaſe, and by decreaſing will approach nearer and nearer to the Ratio of the Velocities at G and K, and will finally vaniſh at the ſame time, and in the proportion of thoſe Velocities, that is, in the Ratio of GN to KN. Conſequently in the Equation now form'd, if we ſuppoſe Y and X to decreaſe continually, and at laſt to vaniſh, that we may obtain their ultimate Ratio; we ſhall thereby obtain the Ratio of GN to KN. But when Y and X vaniſh, or when the point F coincides with G, and I with H, then it will be EG $= y$, and HK $= x$; ſo that we ſhall have $\dot{y} : \dot{x} ::$ GM : KN. And hence we ſhall obtain a Fluxional Equation, which will always exhibit the relation of the Fluxions, or Velocities, belonging to the given Algebraical or Fluential Equation.

Thus, for Example, if EF $= y$, and HI $= x$, and the indefinite Lines y and x are ſuppoſed to increaſe at ſuch a rate, as that their relation may always be expreſs'd by this Equation $x^3 - ax^2 + axy - y^3 = 0$; then making FG $=$ Y, and IK $=$ X, by ſubſtituting $y + \mathrm{Y}$ for y, and $x + \mathrm{X}$ for x, and reducing the Equation that will ariſe, (ſee before, pag. 255.) we ſhall have $3x^2\mathrm{X} + 3x\mathrm{X}^2 + \mathrm{X}^3 - 2ax\mathrm{X} - a\mathrm{X}^2 + ax\mathrm{Y} + a\mathrm{X}y + a\mathrm{XY} - 3y^2\mathrm{Y} - 3y\mathrm{Y}^2 - \mathrm{Y}^3 = 0$, which may be thus expreſs'd in an Analogy, $\mathrm{Y} : \mathrm{X} :: 3x^2 - 2ax + ay + 3x\mathrm{X} + \mathrm{X}^2 - a\mathrm{X} : 3y^2 - ax - a\mathrm{X} + 3y\mathrm{Y} + \mathrm{Y}^2$. This Analogy, when Y and X are vaniſhing quantities, or their ultimate Ratio, will become $\mathrm{Y} : \mathrm{X} :: 3x^2 - 2ax + ay : 3y^2 - ax$. And becauſe it is then $\mathrm{Y} : \mathrm{X} :: \mathrm{GN} : \mathrm{KN} :: \dot{y} : \dot{x}$, it will be $\dot{y} : \dot{x} :: 3x^2 - 2ax + ay : 3y^2 - ax$. Which gives the proportion of the Fluxions. And the like in all other caſes. Q. E. I.

We might alſo lay a foundation for theſe Speculations in the following manner. Let ABCDEF, &c. be the Periphery of a Polygon, or any part of it, and let the Sides AB, BC, CD, DE, &c. be of any magnitude whatever. In the ſame Plane, and at any diſtance, draw the two parallel Lines $\beta\zeta$, and bf, to which continue the right Lines AB$b\beta$, BC$c\gamma$, CD$d\delta$, DE$e\varepsilon$, &c. meeting the parallels as in the figure. Now if we ſup-



poſe two moving points, or bodies, to be at *β* and *b*, and to move in the ſame time to *γ* and *c*, with any equable Velocities; thoſe Velocities will be to each other as *βγ* and *bc*, that is, becauſe of the parallels, as *β*B and *b*B. Let them ſet out again from *γ* and *c*, and arrive at the ſame time at *δ* and *d*, with any equable Velocities; thoſe Velocities will be as *γδ* and *cd*, that is, as *γ*C and *c*C. Let them depart again from *δ* and *d*, and arrive in the ſame time at *ε* and *e*, with any equable Velocities; thoſe Velocities will be as *δε* and *de*, that is, as *δ*D and *d*D. And it will be the ſame thing every where, how many ſoever, and how ſmall ſoever, the Sides of the Polygon may be. Let their number be increaſed, and their magnitude be diminiſh'd *in infinitum*, and then the Periphery of the Polygon will continually approach towards a Curve-line, to which the Lines AB*bβ*, BC*cγ*, CD*dδ*, &c. will become Tangents; as alſo the Motions may be conceived to degenerate into ſuch as are accelerated or retarded continually. Then in any two points, ſuppoſe *δ* and *d*, where the deſcribing points are found at the ſame time, their Velocities (or Fluxions) will be as the Segments of the reſpective Tangents *δ*D and *d*D; and the Lines *βδ* and *bd*, intercepted by any two Tangents *δ*D and *β*B, will be the contemporaneous Lines, or Fluents. Now from the nature of the Curve being given, or from the property of its Tangents, the contemporaneous Lines may be found, or the relation of the Fluents. And *vice verſâ*, from the Rate of flowing being given, the correſponding Curve may be found.

ANNOTATIONS on Prob. 2.

OR,

The Relation of the Fluxions being given, to find the Relation of the Fluents.

SECT. I. *A particular Solution; with a preparation for the general Solution, by which it is diſtributed into three Caſes.*

1, 2. WE are now come to the Solution of the Author's ſecond fundamental Problem, borrow'd from the Science of Rational Mechanicks: Which is, from the Velocities of the Motion at all times given, to find the quantities of the Spaces deſcribed; or to find the Fluents from the given Fluxions. In diſcuſſing which important Problem, there will be occaſion to expatiate ſomething more at large. And firſt it may not be amiſs to take notice, that in the Science of Computation all the Operations are of two kinds, either Compoſitive or Reſolutative. The Compoſitive or Synthetic Operations proceed neceſſarily and directly, in computing their ſeveral *quæſita*, and not tentatively or by way of tryal. Such are Addition, Multiplication, Raiſing of Powers, and taking of Fluxions. But the Reſolutative or Analytical Operations, as Subtraction, Diviſion, Extraction of Roots, and finding of Fluents, are forced to proceed indirectly and tentatively, by long deductions, to arrive at their ſeveral *quæſita*; and ſuppoſe or require the contrary Synthetic Operations, to prove and confirm every ſtep of the Proceſs. The Compoſitive Operations, always when the *data* are finite and terminated, and often when they are interminate or infinite, will produce finite concluſions; whereas very often in the Reſolutative Operations, tho' the *data* are in finite Terms, yet the *quæſita* cannot be obtain'd without an infinite Series of Terms. Of this we ſhall ſee frequent Inſtances in the ſubſequent Operation, of returning to the Fluents from the Fluxions given.

The Author's particular Solution of this Problem extends to ſuch caſes only, wherein the Fluxional Equation propoſed either has been, or at leaſt might have been, derived from ſome finite Algebraical Equation, which is now required. Here all the neceſſary Terms being preſent, and no more than what are neceſſary, it will not be difficult, by a Proceſs juſt contrary to the former, to return back again to the original Equation. But it will moſt commonly happen, either if we aſſume a Fluxional Equation at pleaſure, or if we arrive at one as the reſult of ſome Calculation, that ſuch an Equation is to be reſolved, as could not be derived from any previous finite Algebraical Equation, but will have Terms either redundant or deficient; and conſequently the Algebraic Equation required, or its Root, muſt be had by Approximation only, or by an infinite Series. In all which caſes we muſt have recourſe to the general Solution of this Problem, which we ſhall find afterwards.

The Precepts for this particular Solution are theſe. (1.) All ſuch Terms of the given Equation as are multiply'd (ſuppoſe) by $\dot{x}$, muſt be diſpoſed according to the Powers of x, or muſt be made a Number belonging to the Arithmetical Scale whoſe Root is x. (2.) Then they muſt be divided by $\dot{x}$, and multiply'd by x; or $\dot{x}$ muſt be changed into x, by expunging the point. (3.) And laſtly, the Terms muſt be ſeverally divided by the Progreſſion of the Indices of the Powers of x, or by ſome other Arithmetical Progreſſion, as need ſhall require. And the ſame things muſt be repeated for every one of the flowing quantities in the given Equation.

Thus in the Equation $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$, the Terms $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y$ by expunging the points become $3x^3 - 2ax^2 + axy$, which divided by the Progreſſion of the Indices 3, 2, 1, reſpectively, will give $x^3 - ax^2 + axy$. Alſo the Terms $- 3\dot{y}y^2 * + a\dot{y}x$ by expunging the points become $- 3y^3 * + ayx$, which divided by the Progreſſion of the Indices 3, 2, 1, reſpectively, will give $- y^3 * + ayx$. The aggregate of theſe, neglecting the redundant Term ayx, is $x^3 - ax^2 + axy - y^3 = 0$, the Equation required. Where it muſt be noted, that every Term, which occurs more than once, muſt be accounted a redundant Term.

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So if the propoſed Equation were $\overline{m+3}y\dot{x}x^3 - \overline{m+2}ay\dot{x}x^2 + \overline{m+1}ay^2\dot{x}x - my^4\dot{x} - \overline{n+3}x\dot{y}y^3 + \overline{n+1}ax^2\dot{y}y + nx^4\dot{y} - nax^3\dot{y} = 0$, whatever values the general Numbers m and n may acquire; if thoſe Terms in which $\dot{x}$ is found are reduced to the Scale whoſe Root is x, they will ſtand thus: $\overline{m+3}y\dot{x}x^3 - \overline{m+2}ay\dot{x}x^2 + \overline{m+1}ay^2\dot{x}x - my^4\dot{x}$; or expunging the points they will become $\overline{m+3}yx^4 - \overline{m+2}ayx^3 + \overline{m+1}ay^2x^2 - my^4x$. Theſe being divided reſpectively by the Arithmetical Progreſſion $m+3$, $m+2$, $m+1$, m, will give the Terms $yx^4 - ayx^3 + ay^2x^2 - y^4x$. Alſo the Terms in which $\dot{y}$ is found; being reduced to the Scale whoſe Root is y, will ſtand thus: $-\overline{n+3}x\dot{y}y^3 * + \overline{n+1}ax^2\dot{y}y + nx^4\dot{y}$;
$- nax^3\dot{y}$
or expunging the points they will become $-\overline{n+3}xy^4 * + \overline{n+1}ax^2y^2 + nx^4y$.
$- nax^3y$
Theſe being divided reſpectively by the Arithmetical Progreſſion $\overline{n+3}$, $\overline{n+2}$, $\overline{n+1}$, n, will give the Terms $-xy^4 + ax^2y^2 + x^4y - ax^3y$. But theſe Terms, being the ſame as the former, muſt all be conſider'd as redundant, and therefore are to be rejected. So that $yx^4 - ayx^3 + ay^2x^2 - y^4x = 0$, or dividing by yx, the Equation $x^3 - ax^2 + ayx - y^3 = 0$ will ariſe as before.

Thus if we had this Fluxional Equation $may\dot{x}x^{-1} - \overline{m+2}\dot{x}x - nx^2\dot{y}y^{-1} + \overline{n+1}a\dot{y} = 0$, to find the Fluential Equation to which it belongs; the Terms $may\dot{x}x^{-1} * - \overline{m+2}\dot{x}x$, by expunging the points, and dividing by the Terms of the Progreſſion m, $m+1$, $m+2$, will give the Terms $ay - x^2$. Alſo the Terms $-nx^2\dot{y}y^{-1} + \overline{n+1}a\dot{y}$, by expunging the points, and dividing by n, $n+1$, will give the Terms $-x^2 + ay$. Now as theſe are the ſame as the former, they are to be eſteem'd as redundant, and the Equation required will be $ay - x^2 = 0$. And when the given Fluxional Equation is a general one, and adapted to all the forms of the Fluential Equation, as is the caſe of the two laſt Examples; then all the Terms ariſing from the ſecond Operation will be always redundant, ſo that it will be ſufficient to make only one Operation.

Thus if the given Equation were $4\dot{y}y^2 + z^3\dot{y}y^{-1} + 2y\dot{x}x - 3\dot{z}z^2 + 6y\dot{z}z - 2cy\dot{z} = 0$, in which there are found three flowing quantities; the only Term in which $\dot{x}$ is found is $2y\dot{x}x$, in which expunging the point, and then dividing by the Index 2, it will become yx^2. Then the Terms in which $\dot{y}$ is found are $4\dot{y}y^2 + z^3\dot{y}y^{-1}$, which expunging the points become $4y^3 * * + z^3$, and dividing by the Progreſſion 2, 1, 0, — 1, give the Terms $2y^3 - z^3$. Laſtly the Terms in which $\dot{z}$ is found are $- 3\dot{z}z^2 + 6y\dot{z}z - 2cy\dot{z}$, which expunging the points become $- 3z^3 + 6yz^2 - 2cyz$, and dividing by the Progreſſion 3, 2, 1, give the Terms $- z^3 + 3yz^2 - 2cyz$. Now if we collect theſe Terms, and omit the redundant Term $- z^3$, we ſhall have $yx^2 + 2y^3 - z^3 + 3yz^2 - 2cyz = 0$ for the Equation required.

3, 4. But theſe deductions are not to be too much rely'd upon, till they are verify'd by a proof; and we have here a ſure method of proof, whether we have proceeded rightly or not, in returning from the relation of the Fluxions to the relation of the Fluents. For every reſolutative Operation ſhould be proved by its contrary compoſitive Operation. So if the Fluxional Equation $\dot{x}x - \dot{x}y - x\dot{y} + a\dot{y} = 0$ were given, to return to the Equation involving the Fluents; by the foregoing Rule we ſhall firſt have the Terms $\dot{x}x - \dot{x}y$, which by expunging the points will become $x^2 - xy$, and dividing by the Progreſſion 2, 1, will give the Terms $\frac{1}{2}x^2 - xy$. Alſo the Terms, or rather Term, $- x\dot{y} + a\dot{y}$, by expunging the points will become $- xy. + ay$, which are only to be divided by Unity. So that leaving out the redundant Term $- xy$, we ſhall have the Fluential Equation $\frac{1}{2}x^2 - xy + ay = 0$. Now if we take the Fluxions of this Equation, we ſhall find by the foregoing Problem $x\dot{x} - \dot{x}y - x\dot{y} + a\dot{y} = 0$, which being the ſame as the Equation given, we are to conclude our work is true. But if either of the Fluxional Equations $x\dot{x} - \dot{x}y + a\dot{y} = 0$, or $x\dot{x} - x\dot{y} + a\dot{y} = 0$ had been propoſed, tho' by purſuing the foregoing method we ſhould arrive at the Equation $\frac{1}{2}x^2 - xy + ay = 0$, for the relation of the Fluents; yet as this concluſion would not ſtand the teſt of this proof, we muſt reject it as erroneous, and have recourſe to the following general Method; which will give the value of y in either of thoſe Equations by an infinite Series, and therefore for uſe and practice will be the moſt commodious Solution.

5. As Velocities can be compared only with Velocities, and all other quantities with others of the ſame Species only; therefore in every Term of an Equation, the Fluxions muſt always aſcend to the ſame number of Dimenſions, that the homogeneity may not be deſtroy'd. Whenever it happens otherwiſe, 'tis becauſe ſome Fluxion, taken for Unity, is there underſtood, and therefore muſt be ſupply'd when occaſion requires. The Equation $\dot{x}\dot{z} + \dot{x}\dot{y}x - a\dot{z}^2x^2 = 0$, by making $\dot{z} = 1$, may become $\dot{x} + \dot{x}\dot{y}x - ax^2 = 0$, and likewiſe *vice verſâ*. And as this Equation virtually involves three variable

quantities, it will require another Equation, either Fluential or Fluxional, for a compleat determination, as has been already obſerved. So as the Equation $\dot{y}x = \dot{x}yy$, by putting $\dot{x} = 1$ becomes $\dot{y}x = yy$; in like manner this Equation requires and ſuppoſes the other.

6, 7, 8, 9, 10, 11. Here we are taught ſome uſeful Reductions, in order to prepare the Equation for Solution. As when the Equation contains only two flowing Quantities with their Fluxions, the ratio of the Fluxions may always be reduced to ſimple Algebraic Terms. The Antecedent of the Ratio, or its Fluent, will be the quantity to be extracted; and the Conſequent, for the greater ſimplicity, may be made Unity. Thus the Equation $2\dot{x} + 2x\dot{x} - y\dot{x} - \dot{y} = 0$ is reduced to this, $\frac{\dot{y}}{\dot{x}} = 2 + 2x - y$, or making $\dot{x} = 1$, 'tis $\dot{y} = 2 + 2x - y$. So the Equation $\dot{y}a - \dot{y}x - \dot{x}a + \dot{x}x - \dot{x}y = 0$, making $\dot{x} = 1$, will become $\dot{y} = (\frac{a - x + y}{a - x} = 1 + \frac{y}{a - x} =)\ 1 + \frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$, &c. by Diviſion. But we may apply the particular Solution to this Example, by which we ſhall have $\frac{1}{2}x^2 - xy - ax + ay = 0$, and thence $y = \frac{ax - \frac{1}{2}x^2}{a - x}$. Thus the Equation $\dot{y}\dot{y} = \dot{x}\dot{y} + \dot{x}\dot{x}xx$, making $\dot{x} = 1$, becomes $\dot{y}\dot{y} = \dot{y} + xx$, and extracting the ſquare-root, 'tis $\dot{y} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + xx} = \frac{1}{2} \pm$ the Series $\frac{1}{2} + x^2 - x^4 + 2x^6 - 5x^8 + 14x^{10}$, &c. that is, either $\dot{y} = 1 + x^2 - x^4 + 2x^6 - 5x^8 + 14x^{10}$, &c. or $\dot{y} = -x^2 + x^4 - 2x^6 + 5x^8 - 14x^{10}$, &c. Again, the Equation $\dot{y}^3 + ax\dot{x}^2\dot{y} + a^2\dot{x}^2\dot{y} - x^3\dot{x}^3 - 2\dot{x}^3a^3 = 0$, putting $\dot{x} = 1$, becomes $\dot{y}^3 + ax\dot{y} + a^2\dot{y} - x^3 - 2a^3 = 0$. Now an affected Cubic Equation of this form has been reſolved before, (pag. 12.) by which we ſhall have $\dot{y} = a - \frac{1}{4}x + \frac{xx}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$, &c.

12. For the ſake of perſpicuity, and to fix the Imagination, our Author here introduces a diſtinction of Fluents and Fluxions into *Relate* and *Correlate*. The Correlate is that flowing Quantity which he ſuppoſes to flow equably, which is given, or may be aſſumed, at any point of time, as the known meaſure or ſtandard, to which the Relate Quantity may be always compared. It may therefore very properly denote Time; and its Velocity or Fluxion, being an uniform and conſtant quantity, may be made the Fluxional Unit, or the known meaſure of the Fluxion (or of the rate of flowing) of the Relate Quantity. The Relate Quantity, (or Quantities if ſeve-

ral are concern'd,) is that which is ſuppos'd to flow inequably, with any degrees of acceleration or retardation; and ts inequability may be meaſured, or reduced as it were to equability, by conſtantly comparing it with its correſponding Correlate or equable Quantity. This therefore is the Quantity to be found by the Problem, or whoſe Root is to be extracted from the given Equation. And it may be conceived as a Space deſcribed by the inequable Velocity of a Body or Point in motion, while the equable Quantity, or the Correlate, repreſents or meaſures the time of deſcription. This may be illuſtrated by our common Mathematical Tables, of Logarithms, Sines, Tangents, Secants, &c. In the Table of Logarithms, for inſtance, the Numbers are the Correlate Quantity, as proceeding equably, or by equal differences, while their Logarithms, as a Relate Quantity, proceed inequably and by unequal differences. And this reſemblance would more nearly obtain, if we ſhould ſuppoſe infinite other Numbers and their Logarithms to be interpolated, (if that infinite Number be every where the ſame,) ſo as that in a manner they may become continuous. So the Arches or Angles may be conſider'd as the Correlate Quantity, becauſe they proceed by equal differences, while the Sines, Tangents, Secants, &c. are as ſo many Relate Quantities, whoſe rate of increaſe is exhibited by the Tables.

13, 14, 15, 16, 17. This Diſtribution of Equations into Orders, or Claſſes, according to the number of the flowing Quantities and their Fluxions, tho' it be not of abſolute neceſſity for the Solution, may yet ſerve to make it more expedite and methodical, and may ſupply us with convenient places to reſt at.

SECT. II. *Solution of the firſt Caſe of Equations.*

18, 19, 20, 21, 22, 23. THE firſt Caſe of Equations is, when the Quantity $\frac{\dot{y}}{\dot{x}}$, or what ſupplies its place, can always be found in Terms compoſed of the Powers of x, and known Quantities or Numbers. Theſe Terms are to be multiply'd by x, and to be divided by the Index of x in each Term, which will then exhibit the Value of y. Thus in the Equation $\dot{y}^2 = \dot{x}\dot{y} + \dot{x}^2x^2$; it has been found that $\frac{\dot{y}}{\dot{x}} = 1 + x^2 - x^4 + 2x^6 - 5x^8 + 14x^{10}$, &c. Therefore $\frac{\dot{y}x}{\dot{x}} = x + x^3 - x^5 + 2x^7 - 5x^9 + 14x^{11}$, &c. and conſequently $y = x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7 - \frac{5}{9}x^9 + \frac{14}{11}x^{11}$, &c. as may eaſily be proved by the direct Method.

But this, and the like Equations, may be reſolved more readily by a Method form'd in imitation of ſome of the foregoing Analyſes, after this manner. In the given Equation make $\dot{x} = 1$; then it will be $\dot{y}^2 = \dot{y} + x^2$, which is thus reſolved:

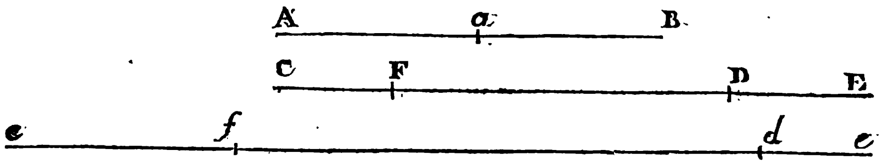
$$\left.\begin{array}{r}\dot{y}\\ -\dot{y}^2\end{array}\right\}\begin{array}{l} = -x^2 + x^4 - 2x^6 + 5x^8, \text{ \&c.}\\ \text{- - - - - - - - } x^4 + 2x^6 - 5x^8, \text{ \&c.}\end{array}$$

Make $-x^2$ the firſt Term of $\dot{y}$; then will $-x^4$ be the firſt Term of $-\dot{y}^2$, which is to be put with a contrary Sign for the ſecond Term of $\dot{y}$. Then by ſquaring, $+2x^6$ will be the ſecond Term of $-\dot{y}^2$, and $-2x^6$ will be the third Term of $\dot{y}$. Therefore $-5x^8$ will be the third Term of $-\dot{y}^2$, and $+5x^8$ will be the fourth Term of $\dot{y}$; and ſo on. Therefore taking the Fluents, $y = -\frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{2}{7}x^7 + \frac{5}{9}x^9$, &c. which will be one Root of the Equation. And if we ſubtract this from x, we ſhall have $y = x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7 - \frac{5}{9}x^9$, &c. for the other Root.

So if $\frac{\dot{y}}{\dot{x}} = a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2}$, &c. that is, if $\frac{\dot{y}x}{\dot{x}} = ax - \frac{1}{4}x^2 + \frac{x^3}{64a} + \frac{131x^4}{512a^2}$, &c. then $y = ax - \frac{1}{8}x^2 + \frac{x^3}{192a} + \frac{131x^4}{2048a^2}$, &c. If $\frac{\dot{y}}{\dot{x}} = \frac{1}{x^3} - \frac{1}{x^2} + \frac{a}{x^{\frac{1}{2}}} - x^{\frac{1}{2}} + x^{\frac{3}{2}}$, &c. or $\frac{\dot{y}x}{\dot{x}} = x^{-2} - x^{-1} + ax^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}}$, &c. then $y = -\frac{1}{2}x^{-2} + x^{-1} + 2ax^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}$, &c. If $\frac{\dot{x}}{\dot{y}} = \frac{2b^2c}{\sqrt{ay^3}} + \frac{3y^2}{a+b} + \sqrt{by + cy}$, or $\frac{\dot{x}y}{\dot{y}} = \frac{2b^2c\overline{y}^{-\frac{1}{2}}}{a^{\frac{1}{2}}} + \frac{3y^3}{a+b} + y^{\frac{3}{2}}\sqrt{b+c}$, then $x = -\frac{4b^2c}{\sqrt{ay}} + \frac{y^3}{a+b} + \frac{2}{3}y^{\frac{3}{2}}\sqrt{b+c}$. If $\frac{\dot{y}}{\dot{z}} = z^{\frac{2}{3}}$, or $\frac{\dot{y}z}{\dot{z}} = z^{\frac{5}{3}}$, then $y = \frac{3}{5}z^{\frac{5}{3}}$. If $\frac{\dot{y}}{\dot{x}} = \frac{ab}{cx^{\frac{1}{3}}} = \frac{ab}{c}x^{-\frac{1}{3}}$, or $\frac{\dot{y}x}{\dot{x}} = \frac{ab}{c}x^{\frac{2}{3}}$, then $y = \frac{3ab}{2c}x^{\frac{2}{3}}$.

Laſtly, if $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$, or $\frac{\dot{y}x}{\dot{x}} = a = ax^0$; dividing by the Index 0, it will be $y = \frac{a}{0}$, or y is infinite. That this Expreſſion, or value of y, muſt be infinite, is very plain. For as 0 is a vaniſhing quantity, or leſs than any aſſignable quantity, its Reciprocal $\frac{1}{0}$ or $\frac{a}{0}$ muſt be bigger than any aſſignable quantity, that is, infinite.

Now that this quantity ought to be infinite, may be thus proved. In the Equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$, let AB repreſent the conſtant quantity *a*, and in CE let a point move equably from C towards E, and deſcribe the Line CDE; of which let any indefinite part CD be *x*, and its equable Velocity in D, (and every where elſe,) is repreſented



by $\dot{x}$. Alſo let a point move from a diſtant point *c* along the Line *cde*, with an inequable Velocity, and let the Line deſcribed in the ſame time, or the indefinite part of it *cd*, be call'd *y*, and let the Velocity in *d* be call'd $\dot{y}$. The Equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$ muſt always obtain, whatever the contemporaneous values of *x* and *y* may be; or in the whole Motion the conſtant Line AB (*a*) muſt be to the variable Line CD (*x*), as the Velocity in *d* ($\dot{y}$) is to the Velocity in D ($\dot{x}$). But at the beginning of the Motion, or when CD (*x*) was indefinitely little, as the ratio of AB to CD was then greater than any aſſignable ratio, ſo alſo was the ratio $\frac{\dot{y}}{\dot{x}}$ of the Velocities, or the Velocity $\dot{y}$ was infinitely greater than the Velocity $\dot{x}$. But an infinite Velocity muſt deſcribe an infinite Space in a finite time, or the point *c* is at an infinite diſtance from the point *d*, that is, *y* is an infinite quantity.

24, 25. But to avoid ſuch infinite Expreſſions, from whence we can conclude nothing; we are at liberty to change the initial points of the Fluents, by which their Rate of flowing, (the only thing to be here regarded,) will not at all be affected. Thus in the foregoing Figure, we ſuppoſed the points D and *d* to be ſuch, as limited the contemporaneous Fluents, or in which the two deſcribing points were found at the ſame time. Let F and *f* be any other two ſuch points, and then the finite Line CF $= b$ will be contemporaneous to, or will correſpond with, the infinite Line *cf* $= c$; and FD, which may be made the new *x*, will correſpond to *fd*, which will be the new *y*. So that in the given Equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$, inſtead of

x we may write $b+x$, and we ſhall have $\frac{\dot{y}}{\dot{x}} = \frac{a}{b+x}$, and then by Multiplication and Diviſion it is $\frac{\dot{y}x}{\dot{x}} = \left(\frac{ax}{b+x} = \right) \frac{ax}{b} - \frac{ax^2}{b^2} + \frac{ax^3}{b^3} - \frac{ax^4}{b^4}$, &c. and therefore $y = \frac{ax}{b} - \frac{ax^2}{2b^2} + \frac{ax^3}{3b^3} - \frac{ax^4}{4b^4}$, &c.

26. So if $\frac{\dot{y}}{\dot{x}} = \frac{2}{x} + 3 - xx$, becauſe of the Term $\frac{2}{x}$, which would give an infinite value for y, we may write $1+x$ inſtead of x, and we ſhall then have $\frac{\dot{y}}{\dot{x}} = \frac{2}{1+x} + 2 - 2x - xx$; or $\frac{\dot{y}x}{\dot{x}} = \frac{2x}{1+x} + 2x - 2x^2 - x^3$, or by Diviſion $\frac{\dot{y}x}{\dot{x}} = 4x - 4x^2 + x^3 - 2x^4 + 2x^5$, &c. and therefore $y = 4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{2}{5}x^5$, &c.

Or the Equation $\frac{\dot{y}}{\dot{x}} = \frac{2}{1+x} + 2 - 2x - x^2$, that is $\dot{y} + x\dot{y} = 4 - 3x^2 - x^3$, may be thus reſolved:

$$\begin{array}{rlllllll}
\left.\begin{array}{r}\dot{y}\\ \\ +x\dot{y}\end{array}\right\} & = 4 & * & -3x^2 & -x^3 & & \\
 & & -4x & +4x^2 & -x^3 & +2x^4, & \&c. \\
 & \text{- - - -} & +4x & -4x^2 & +x^3 & -2x^4, & \&c. \\
\dot{y} & = 4 & -4x & +x^2 & -2x^3 & +2x^4, & \&c. \\
y & = 4x & -2x^2 & +\frac{5}{3}x^3 & -\frac{1}{2}x^4 & +\frac{2}{5}x^5, & \&c.
\end{array}$$

Make 4 the firſt Term of $\dot{y}$, then $4x$ will be the firſt Term of $x\dot{y}$, and conſequently $-4x$ will be the ſecond Term of $\dot{y}$. Then $-4x^2$ will be the ſecond Term of $x\dot{y}$, and therefore $+4x^2 - 3x^2$, or x^2, will be the third Term of $\dot{y}$; and ſo on.

27. So if $\frac{\dot{y}}{\dot{x}} = x^{-\frac{1}{2}} + x^{-1} - x^{\frac{1}{2}}$, becauſe of the Term x^{-1} change x into $1-x$, then $\frac{\dot{y}}{\dot{x}} = \frac{1}{\sqrt{1-x}} + \frac{1}{1-x} - \sqrt{1-x}$. But by the foregoing Methods of Reduction 'tis $\frac{1}{1-x} = 1 + x + x^2 + x^3$, &c. and $\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$, &c. and $\frac{1}{\sqrt{1-x}} = \frac{1}{1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3, \&c.} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{8}x^3$; &c. Therefore collecting theſe according to their Signs, 'tis $\frac{\dot{y}}{\dot{x}} = 1 + 2x + \frac{3}{2}x^2 + \frac{27}{16}x^3$, &c. that is $\frac{\dot{y}x}{\dot{x}} = x + 2x^2 + \frac{3}{2}x^3 + \frac{27}{16}x^4$, &c. and therefore $y = x + x^2 + \frac{1}{2}x^3 + \frac{27}{64}x^4$, &c.

28. So if the given Equation were $\frac{\dot{y}}{\dot{x}} = \frac{c^2x}{c^3 - 3c^2x + 3cx^2 - x^3} = \frac{c^2x}{\overline{c-x}|^3}$; change the beginning of x, that is, inſtead of x write

$c - x$,

$c - x$, then $\frac{\dot{y}}{\dot{x}} = \frac{c^3 - c^2x}{x^3} = c^3x^{-3} - c^2x^{-2}$, or $\frac{\dot{y}x}{\dot{x}} = c^3x^{-2} - c^2x^{-1}$, and therefore $y = -\frac{1}{2}c^3x^{-2} + c^2x^{-1}$.

SECT. III. *Solution of the ſecond Caſe of Equations.*

29, 30. EQuations belonging to this ſecond caſe are thoſe, wherein the two Fluents and their Fluxions, ſuppoſe x and y, $\dot{x}$ and $\dot{y}$, or any Powers of them, are promiſcuouſly involved. As our Author's Analyſes are very intelligible, and ſeem to want but little explication, I ſhall endeavour to reſolve his Examples in ſomething an eaſier and ſimpler manner, than is done here; by applying to them his own artifice of the Parallelogram, when needful, or the properties of a combined Arithmetical Progreſſion *in plano*, as explain'd before: As alſo the Methods before made uſe of, in the Solution of affected Equations.

31. The Equation $\dot{y}ax - \dot{x}xy - aa\dot{x} = 0$ by a due Reduction becomes $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{x}$, in which, becauſe of the Term $\frac{a}{x}$ there is occaſion for a Tranſmutation, or to change the beginning of the Correlate Quantity x. Aſſuming therefore the conſtant quantity b, we may put $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{b+x}$, whence by Diviſion will be had $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{b} - \frac{ax}{b^2} + \frac{ax^2}{b^3} - \frac{ax^3}{b^4}$, &c. which Equation is then prepared for the Author's Method of Solution.

But without this previous Reduction to an infinite Series, and the Reſolution of an infinite Equation conſequent thereon, we may perform the Solution thus, in a general manner. The given Equation is now $\frac{\dot{y}}{\dot{x}} = \frac{y}{a} + \frac{a}{b+x}$, or putting $\dot{x} = 1$, it is $ab\dot{y} + ax\dot{y} = by + yx + a^2$, which may be thus reſolved:

$$\left.\begin{array}{r} ab\dot{y} \\ +ax\dot{y} \\ -by \\ -xy \end{array}\right\} \begin{array}{llll} = a^2 + \frac{ab-a^2}{b}x & + \frac{2a^2+b^2-ab}{2b^2}x^2 & + \frac{b^3+2a^2b-ab^2-6a^3}{6ab^3}x^3, & \&c. \\ ----+ \frac{a^2}{b}x & + \frac{ab-aa}{bb}x^2 & + \frac{2a^2+b^2-ab}{2b^3}x^3, & \&c. \\ ----- ax & + \frac{a-b}{2b}x^2 & + \frac{ab-2a^2-b^2}{6ab^2}x^3, & \&c. \\ ----------- & \frac{a}{b}x^2 & + \frac{a-b}{2b^2}x^3, & \&c. \end{array}$$

$$\dot{y} = \frac{a}{b} + \frac{b-a}{b^2}x + \frac{2a^2+b^2-ab}{2ab^3}x^2 + \frac{b^3+2a^2b-ab^2-6a^3}{6a^2b^4}x^3, \ \&c.$$

$$y = \frac{a}{b}x + \frac{b-a}{2b^2}x^2 + \frac{2a^2+b^2-ab}{6ab^3}x^3 + \frac{b^3+2a^2b-ab^2-6a^3}{24a^2b^4}x^4, \ \&c.$$

Diſpoſing the Terms as you ſee is done here, make a^2 the firſt Term of $ab\dot{y}$, then $\frac{a}{b}$ will be the firſt Term of $\dot{y}$, and thence $\frac{a}{b}x$ will be the firſt Term of y. So that $\frac{a^2}{b}x$ will be the firſt Term of $ax\dot{y}$, and $-ax$ will be the firſt Term of $-by$. Theſe two together, or $\frac{a^2}{b}x - ax = \frac{a^2-ab}{b}x$, with a contrary Sign, muſt be put down for the ſecond Term of $ab\dot{y}$. Therefore the ſecond Term of $\dot{y}$ will be $\frac{b-a}{b^2}x$, and the like Term of y will be $\frac{b-a}{2b^2}x^2$. Then the ſecond Term of $ax\dot{y}$ will be $\frac{ab-a^2}{b^2}x^2$, and the ſecond Term of $-by$ will be $\frac{a-b}{2b}x^2$, and the firſt Term of $-xy$ will be $-\frac{a}{b}x^2$. Theſe three together make $\frac{ab-2a^2-b^2}{2b^2}x^2$, which with a contrary Sign muſt be made the third Term of $ab\dot{y}$. Therefore the third Term of $\dot{y}$ will be $\frac{2a^2+b^2-ab}{2ab^3}x^2$, and the third Term of y will be $\frac{2a^2+b^2-ab}{6ab^3}x^3$. And ſo on. Here in a particular caſe if we make $b = a$, we ſhall have the ſimple Series $y = x * + \frac{x^3}{3a^2} - \frac{x^4}{6a^3}$, &c.

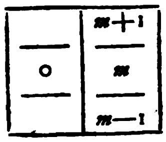
Or if we would have a deſcending Series for the Root y of this Equation, we may proceed as follows:

$$\begin{array}{rlllll}
\left.\begin{array}{r} -xy \\ -by \\ +ax\dot{y} \\ +ab\dot{y} \end{array}\right\} & \begin{array}{l} = a^2 \\ \text{- - -} \\ \text{- - -} \\ \text{- - - - -} \end{array} & \begin{array}{l} -\overline{a+b}\times a^2x^{-1} \\ +a^2bx^{-1} \\ +a^3x^{-1} \\ \text{- - - - - - - -} \end{array} & \begin{array}{l} +\overline{2a^2+2ab+b^2}\times a^2x^{-2}, \\ -\overline{a+b}\times a^2bx^{-2}, \\ -\overline{a+b}\times 2a^3x^{-2}, \\ +a^3bx^{-2}, \end{array} & \begin{array}{l} \&c. \\ \&c. \\ \&c. \\ \&c. \end{array} \\
y & = -a^2x^{-1} & +\overline{a+b}\times a^2x^{-2} & -\overline{2a^2+2ab+b^2}\times a^2x^{-3}, & \&c. \\
\dot{y} & = a^2x^{-2} & -\overline{a+b}\times 2a^2x^{-3}, \&c. & &
\end{array}$$

Diſpoſe the Terms as you ſec, and make a^2 the firſt Term of the Series $-xy$; then will $-\frac{a^2}{x}$ be the firſt Term of y, and a^2x^{-2} will be the firſt Term of $\dot{y}$. Then will $+a^2bx^{-1}$ be the firſt Term of $-by$, and a^3x^{-1} will be the firſt Term of $ax\dot{y}$, which together make $\overline{a+b}\times a^2x^{-1}$; this therefore with a contrary Sign muſt be the ſecond Term of $-xy$. Then the ſecond Term of y will be $\overline{a+b}\times a^2x^{-2}$, and the ſecond Term of $\dot{y}$ will be $-\overline{a+b}\times 2a^2x^{-3}$. Therefore the ſecond Term of $-by$ will be $-\overline{a+b}\times a^2bx^{-2}$, and the ſecond Term of $ax\dot{y}$ will be $-\overline{a+b}\times 2a^3x^{-2}$, and the firſt Term of $ab\dot{y}$ will be a^3bx^{-2}; which three together make $-\overline{2a^2+2ab+b^2}\times a^2x^{-2}$. This with a contrary Sign muſt be the third Term of $-xy$, which will give $-\overline{2a^2+2ab+b^2}\times a^2x^{-3}$ for the third Term of y; and ſo on. Here if we make $b=a$, then $y= -\frac{a^2}{x}+\frac{2a^3}{x^2}-\frac{5a^4}{x^3}$, &c.

And theſe are all the Series, by which the value of y can be exhibited in this Equation, as may be proved by the Parallelogram. For that Method may be extended to theſe Fluxional Equations, as well as to Algebraical or Fluential Equations. To reduce theſe Equations within the Limits of that Rule, we are to conſider, that as Ax^m may repreſent the initial Term of the Root y, in both theſe kinds of Equations, or becauſe it may be $y=Ax^m$, &c. ſo in Fluxional Equations (making $\dot{x}=1$, we ſhall have alſo $\dot{y}=mAx^{m-1}$, &c. or writing y for Ax^m, &c. 'tis $\dot{y}=myx^{-1}$, &c. So that in every Term of the given Equation, in which $\dot{y}$ occurs, or the Fluxion of the Relate Quantity, we may conceive it to take away one Dimenſion from the Correlate Quantity, ſuppoſe x, and to add it to the Relate Quantity, ſuppoſe y; according to which Reduction we may inſert the Terms in the Parallelogram. And we are to make a like Reduction for all the Powers of the Fluxion of the Relate Quantity. This will bring all Fluxional Equations to the Caſe of Algebraic Equations, the Reſolution of which has been ſo amply treated of before.

Thus in the preſent Equation $ab\dot{y}+ax\dot{y}=by+yx+aa$, the Terms muſt be inſerted in the Parallelogram, as if yx^{-1} were ſubſtituted inſtead of $\dot{y}$; ſo that the Indices will ſtand as in the Margin, and the Ruler will give only two Caſes of external Terms. Or rather, if we would reduce this Equation to the form of a double Arithmetical Scale, as explain'd before, we ſhould have it in this form. Here in the firſt Column are contain'd thoſe Terms which have y of one Dimenſion, or what is equivalent to it. In the ſecond Column is $-a^2$, or y of no Dimenſions. Alſo in the firſt Line is $-xy$, or ſuch Terms in which x is of one Dimenſion. In the ſecond Line are the Terms $\left.\begin{matrix}-by\\+ax\dot{y}\end{matrix}\right\}-a^2$, which have no Dimenſions of x, becauſe $+ax\dot{y}$ is regarded as if it were ay. Laſtly, in the third line is $ab\dot{y}$, or the Term in which x is of one negative Dimenſion,



$$\left.\begin{matrix}-xy & * \\ \left.\begin{matrix}-by\\+ax\dot{y}\end{matrix}\right\} & -a^2 \\ +ab\dot{y} & *\end{matrix}\right\}=0.$$

Dimenſion, becauſe $+ab\dot{y}$ is conſider'd as if it were $+abx^{-1}\dot{y}$. And theſe Terms being thus diſpos'd, it is plain there can be but two Caſes of external Terms, which we have already diſcuſs'd.

32. If the propoſed Equation be $\frac{\dot{y}}{\dot{x}} = 3y - 2x + \frac{x}{y} - \frac{2y}{xx}$, or making $\dot{x} = 1$, 'tis $-\dot{y} + 3y - 2x + xy^{-1} - 2yx^{-2} = 0$; the Solution of which we ſhall attempt without any preparation, or without any new interpretation of the Quantities. Firſt, the Terms are to be diſpos'd according to a double Arithmetical Scale, the Roots of which are y and x, and then they will ſtand as in the Margin. The Method of doing this with certainty in all caſes is as follows. I obſerve in the Equation there are three powers of y, which are y^1, y^0, and y^{-1}; therefore I place theſe in order at the top of the Table. I obſerve likewiſe that there are four Powers of x, which are x^1, x^0, x^{-1}, and x^{-2}; which I place in order in a Column at the right hand; or it will be enough to conceive this to be done. Then I inſert every Term of the Equation in its proper place, according to its Dimenſions of y and x in that Term; filling up the vacancies with Aſteriſms, to denote the abſence of the Terms belonging to them. The Term $-\dot{y}$ I inſert as if it were $-\dot{y}x^{-1}$, as is explain'd before. Then we may perceive, that if we apply the Ruler to the exterior Terms, we ſhall have three caſes that may produce Series; for the fourth caſe, which is that of direct aſcent or deſcent, is always to be omitted, as never affording any Series. To begin with the deſcending Series, which will ariſe from the two external Terms $-2x$ and $+xy^{-1}$. The Terms are to be diſpos'd, and the Analyſis to be perform'd, as here follows:

$$\left.\begin{array}{c|ccc}
 & y^1 & y^0 & y^{-1} \\
\hline
x^1 & * & -2x & +xy^{-1} \\
x^0 & +3y & * & * \\
x^{-1} & -\dot{y} & * & * \\
x^{-2} & -2yx^{-2} & * & *
\end{array}\right\} = 0.$$

$$\left.\begin{array}{r}
xy^{-1} \\
+3y \\
-\dot{y} \\
-2yx^{-2}
\end{array}\right\}
\begin{array}{l}
= 2x - \frac{3}{2} - \frac{9}{8}x^{-1} - \frac{17}{16}x^{-2}, \&c. \\
{-}{-}{-}{-} + \frac{3}{2} + \frac{9}{8}x^{-1} + \frac{27}{16}x^{-2}, \&c. \\
{-}{-}{-}{-}{-}{-}{-}{-}{-}{-} \quad * \quad + \frac{3}{8}x^{-2}, \&c. \\
{-}{-}{-}{-}{-}{-}{-}{-}{-}{-}{-}{-}{-} - x^{-2}, \&c.
\end{array}$$

$$y = \frac{1}{2} + \frac{3}{8}x^{-1} + \frac{9}{16}x^{-2} + \frac{415}{128}x^{-3}, \&c.$$

Make $xy^{-1} = 2x$, &c. then $y^{-1} = 2$, &c. and by Diviſion $y = \frac{1}{2}$, &c. Therefore $3y = \frac{3}{2}$, &c. and conſequently $xy^{-1} = * - \frac{3}{2}$, &c. or $y^{-1} = * - \frac{3}{2}x^{-1}$, &c. and by Diviſion $y = * + \frac{3}{8}x^{-1}$, &c. Therefore $3y = * \frac{9}{8}x^{-1}$, &c. and conſequently $xy^{-1} = ** - \frac{9}{8}x^{-1}$, &c. So that $y^{-1} = ** - \frac{9}{8}x^{-2}$, &c. and by Diviſion $y = ** + \frac{9}{16}x^{-2}$&c. Then $3y = ** + \frac{27}{16}x^{-2}$, &c. and

$-\dot{y} = * + \frac{3}{8}x^{-2}$, &c. and $-2yx^{-2} = -x^{-2}$, &c. Theſe three together make $+\frac{17}{16}x^{-2}$, and therefore $xy^{-1} = * * * - \frac{17}{16}x^{-2}$, &c. ſo that $y = * * * + \frac{115}{128}x^{-3}$, &c. And ſo on.

Another deſcending Series will ariſe from the two external Terms $+3y$ and $-2x$, which may be thus extracted:

$$\begin{array}{r} 3y \\ +xy^{-1} \\ -\dot{y} \\ -2yx^{-2} \end{array} \left.\vphantom{\begin{array}{r} 3y \\ +xy^{-1} \\ -\dot{y} \\ -2yx^{-2} \end{array}}\right\} \begin{array}{l} = 2x - \frac{5}{6} + \frac{17}{24}x^{-1} - \frac{25}{48}x^{-2}, \text{ \&c.} \\ - - - - + \frac{3}{2} + \frac{5}{8}x^{-1} - \frac{13}{48}x^{-2}, \text{ \&c.} \\ - - - - - \frac{2}{3} \quad * \quad + \frac{17}{72}x^{-2}, \text{ \&c.} \\ - - - - - - - - \frac{4}{3}x^{-1} + \frac{5}{9}x^{-2}, \text{ \&c.} \end{array}$$

$$y = \frac{2}{3}x - \frac{5}{18} + \frac{17}{72}x^{-1} - \frac{25}{144}x^{-2}, \text{ \&c.}$$
$$y^{-1} = \frac{3}{2}x^{-1} + \frac{5}{8}x^{-2} - \frac{13}{48}x^{-3}, \text{ \&c.}$$
$$\dot{y} = \frac{2}{3} \quad * \quad - \frac{17}{72}x^{-2}, \text{ \&c.}$$

Make $3y = 2x$, &c. then $y = \frac{2}{3}x$, &c. and (by Diviſion) $y^{-1} = \frac{3}{2}x^{-1}$, &c. and $xy^{-1} = \frac{3}{2}$, &c. and $-\dot{y} = -\frac{2}{3}$, &c. Therefore $3y = * - \frac{5}{6}$, &c. and $y = * - \frac{5}{18}$, &c. and (by Diviſion) $xy^{-1} = * \frac{5}{8}x^{-1}$, &c. and $-\dot{y} = * \, 0$, &c. and $-2yx^{-2} = -\frac{4}{3}x^{-1}$, &c. Therefore $3y = * * + \frac{17}{24}x^{-1}$, &c. and $y = * * + \frac{17}{72}x^{-1}$, &c. &c.

The aſcending Series in this Equation will ariſe from the two external Terms $-2yx^{-2}$ and xy^{-1}; or multiplying the whole Equation by $-y$, (that one of the external Terms may be clear'd from y,) we ſhall have $\dot{y}y - 3y^2 + 2xy - x + 2y^2x^{-2} = 0$, of which the Reſolution is thus:

$$\begin{array}{r} 2y^2x^{-2} \\ +\dot{y}y \\ +2xy \\ -3y^2 \end{array} \left.\vphantom{\begin{array}{r} 2y^2x^{-2} \\ +\dot{y}y \\ +2xy \\ -3y^2 \end{array}}\right\} \begin{array}{l} = x^{\frac{2}{2}} \quad * \quad - \frac{3}{4}x^{\frac{4}{2}} - \frac{2}{\sqrt{2}}x^{\frac{5}{2}} + \frac{9}{4}x^3, \text{ \&c.} \\ - - - - - + \frac{3}{4}x^2 \quad * \quad - \frac{3}{4}x^3, \text{ \&c.} \\ - - - - - - - - - - - + \frac{2}{\sqrt{2}}x^{\frac{5}{2}} \quad * , \text{ \&c.} \\ - - - - - - - - - - - - - - - \frac{3}{2}x^3, \text{ \&c.} \end{array}$$

$$y = \frac{1}{\sqrt{2}}x^{\frac{3}{2}} \quad * \quad - \frac{3}{8\sqrt{2}}x^{\frac{5}{2}} - \frac{1}{2}x^3 + \frac{135}{128\sqrt{2}}x^{\frac{7}{2}}, \text{ \&c.}$$
$$\dot{y} = \frac{3}{2\sqrt{2}}x^{\frac{1}{2}} \quad * \quad - \frac{15}{16\sqrt{2}}x^{\frac{3}{2}} - \frac{3}{2}x^2, \text{ \&c.}$$

Make $2y^2x^{-2} = x$, &c. then $y^2 = \frac{1}{2}x^3$, &c. and $y = \frac{1}{\sqrt{2}}x^{\frac{3}{2}}$, &c. Here becauſe of the fractional Indices, and that the firſt Term of $+2xy$, or $+\frac{2}{\sqrt{2}}x^{\frac{5}{2}}$, may be afterwards admitted, we muſt take 0 for the ſecond Term of $2y^2x^{-2}$, and therefore for the ſecond Term of y. Then $\dot{y}y = \frac{3}{4}x^2$, &c. and conſequently $2y^2x^{-2} = * * - \frac{3}{4}x^2$, &c. and $y^2 = * * - \frac{3}{8}x^4$, &c. and by extracting the ſquare-root; $y = * * - \frac{3}{8\sqrt{2}}x^{\frac{1}{2}}$, &c. Then $\dot{y}y = * + 0$, &c. and $2xy = \frac{2}{\sqrt{2}}x^{\frac{1}{2}}$, &c. and therefore $2y^2x^{-3} = * * * - \frac{2}{\sqrt{2}}x^{\frac{5}{2}}$, &c. and $y = * * * - \frac{1}{2}x^3$, &c. &c.

33, 34. The Author's Proceſs of Reſolution, in this and the following Examples, is very natural, ſimple, and intelligible; it proceeds *ſeriatim & terminatim*, by paſſing from Series to Series, and by gathering Term after Term, in a kind of circulating manner, of which Method we have had frequent inſtances before. By this means he collects into a Series what he calls the Sum, which Sum is the value of $\frac{\dot{y}}{\dot{x}}$ or of the Ratio of the Fluxions of the Relate and Correlate in the given Equation; and then by the former Problem he obtains the value of y. When I firſt obſerved this Method of Solution, in this Treatiſe of our Author's, I confeſs I was not a little pleaſed; it being nearly the ſame, and differing only in a few circumſtances that are not material, from the Method I had happen'd to fall into ſeveral years before, for the Solution of Algebraical and Fluxional Equations. This Method I have generally purſued in the courſe of this work, and ſhall continue to explain it farther by the following Examples.

The Equation of this Example $1 - 3x + y + x^2 + xy - \dot{y} = 0$ being reduced to the form of a double Arithmetical Scale, will ſtand as here in the Margin; and the Ruler will diſcover two caſes to be try'd, of which one may give us an aſcending, and the other a deſcending Series for the Root y. And firſt for the aſcending Series.

$$\begin{array}{c|cc} & y^1 & y^0 \\ \hline x^2 & * & + x^2 \\ x^1 & + xy & - 3x \\ x^0 & + y & + 1 \\ x^{-1} & - \dot{y} & * \end{array} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\\a\end{array}}\right\} = 0.$$

$$\begin{array}{rl} \left.\begin{array}{r} \dot{y} \\ \\ -y \\ -xy \end{array}\right\} & \begin{array}{l} = 1 - 3x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{2}{15}x^5, \text{ \&c.} \\ \qquad + x \quad * \\ \text{- - -} - x + x^2 - \frac{1}{3}x^3 + \frac{1}{6}x^4 - \frac{1}{30}x^5, \text{ \&c.} \\ \text{- - - - - - -} - x^2 + x^3 - \frac{1}{3}x^4 + \frac{1}{6}x^5, \text{ \&c.} \end{array} \\ \dot{y} & = 1 - 2x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{2}{15}x^5, \text{ \&c.} \\ y & = x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6, \text{ \&c.} \end{array}$$

The Terms being diſpoſed as you ſee, make $\dot{y} = 1$, &c. then $y = x$, &c. Therefore $-y = -x$, &c. the Sign of which Term being changed, it will be $\dot{y} = * + x - 3x$, &c. $= * - 2x$, &c.

and

and therefore $y = * - x^2$, &c. Then $-y = * + x^2$, &c. and $-xy = -x^2$, &c. theſe deſtroying each other, 'tis $\dot{y} = * * + x^2$, &c. and therefore $y = * * + \frac{1}{3}x^3$, &c. Then $-y = * * - \frac{1}{3}x^3$, &c. and $-xy = * + x^3$, &c. it will be $\dot{y} = * * * - \frac{2}{3}x^3$, &c. and therefore $y = * * * - \frac{1}{6}x^4$, &c. &c.

The Analyſis in the ſecond caſe will be thus:

$$\begin{array}{r|lllllll}
-xy & = x^2 - 3x + 1 & & & & & & \\
 & \quad - x & +5 & -6x^{-1} & & * & +12x^{-3}, & \&c. \\
-y & ----+ x & -4 & +6x^{-1} & -6x^{-2} & & *, & \&c. \\
+\dot{y} & ---------- & 1 & * & +6x^{-2} & & -12x^{-3}, & \&c. \\
\hline
 & y = -x & +4 & -6x^{-1} & +6x^{-2} & * & -12x^{-4}, & \&c.
\end{array}$$

Make $-xy = x^2$, &c. then $y = -x$, &c. Therefore $-y = x$, &c. and changing the Sign, 'tis $-xy = * - x - 3x$, &c. $= * - 4x$, &c. and therefore $y = * + 4$, &c. Then $-y = * - 4$, &c. and $\dot{y} = -1$, &c. and changing the Signs, 'tis $-xy = * * + 5 + 1$, &c. $= * * + 6$, &c. and $y = * * - 6x^{-1}$, &c. &c.

35, 36. If the given Equation were $\frac{\dot{y}}{\dot{x}} = 1 + \frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$, &c. its Reſolution may be thus perform'd:

$$\begin{array}{r|llllll}
\dot{y} & = 1 & +\frac{x}{a} & +\frac{3x^2}{2a^2} & +\frac{2x^3}{a^3} & +\frac{5x^4}{2a^4}, & \&c. \\
-\frac{y}{a} & ---- & -\frac{x}{a} & -\frac{x^2}{2a^2} & -\frac{x^3}{2a^3} & -\frac{x^4}{2a^4}, & \&c. \\
-\frac{xy}{a^2} & -------- & & -\frac{x^2}{a^2} & -\frac{x^3}{2a^3} & -\frac{x^4}{2a^4}, & \&c. \\
-\frac{x^2y}{a^3} & ------------ & & & -\frac{x^3}{a^3} & -\frac{x^4}{2a^4}, & \&c. \\
-\frac{x^3y}{a^4} & ---------------- & & & & -\frac{x^4}{a^4}, & \&c. \\
\&c. & & & & & & \\
\hline
 & y = x & +\frac{x^2}{2a} & +\frac{x^3}{2a^2} & +\frac{x^4}{2a^3} & +\frac{x^5}{2a^4}, & \&c.
\end{array}$$

Make $\dot{y} = 1$, &c. then $y = x$, &c. Therefore $-\frac{y}{a} = -\frac{x}{a}$, &c. and $\dot{y} = * + \frac{x}{a}$, &c. and therefore $y = * + \frac{x^2}{2a}$, &c. Then $-\frac{y}{a} = * - \frac{x^2}{2a^2}$, &c. and $-\frac{xy}{a^2} = -\frac{x^2}{a^2}$, &c. and therefore $\dot{y} = * * + \frac{3x^2}{2a^2}$, &c. and $y = * * + \frac{x^3}{2a^2}$, &c. And ſo on.

Now in this Example, becauſe the Series $\frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$, &c. is equal to $\frac{y}{a-x}$, it will be $\dot{y} = \frac{y}{a-x} + 1$, or $a\dot{y} - x\dot{y} = y + a - x$, that is, $y\dot{x} + a\dot{x} - x\dot{x} - a\dot{y} + x\dot{y} = 0$; which Equation, by the particular Solution before deliver'd, will give the relation of the Fluents $yx - ay + ax - \frac{1}{2}x^2 = 0$. Hence $y = \frac{ax - \frac{1}{2}xx}{a-x}$, and by Diviſion $y = x + \frac{x^2}{2a} + \frac{x^3}{2a^2} + \frac{x^4}{2a^3}$, &c. as found above.

37. The Equation of this Example being tabulated, or reduced to a double Arithmetical Scale, will ſtand as here in the Margin. Where it may be obſerved, that becauſe of the Series proceeding both ways *ad infinitum*, there can be but one caſe of exterior Terms, of which the Solution here follows:

$$\begin{array}{c|cccccc}
 & y^0 & & y^1 & y^2 & y^3 & y^4, \&c. \\
\hline
x^{-1} & * & - & \dot{y} & * & & \\
x^0 & * & & * & +y^2 & +y^3 & +y^4, \&c. \\
x^1 & -3x & + & 3xy & -xy^2 & -xy^3 & -xy^4, \&c. \\
x^2 & -6x^2 & + & 6x^2y & * & * & * \\
x^3 & -8x^3 & + & 8x^3y & * & * & * \\
x^4 & -10x^4 & + & 10x^4y & * & * & * \\
x^5 & -12x^5 & + & 12x^5y & * & * & * \\
x^6 & -14x^6 & & \&c. & & & \\
\&c. & \&c. & & & & &
\end{array}\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\} = 0.$$

$$\left.\begin{array}{r}
\dot{y} \\
\\
-3xy \\
-6x^2y \\
-y^2 \\
-8x^3y \\
+xy^2 \\
-10x^4y \\
-y^3 \\
\&c.
\end{array}\right\}
\begin{array}{l}
= -3x - 6x^2 - 8x^3 - 10x^4 - 12x^5 - 14x^6, \&c. \\
\qquad\qquad\qquad - \frac{9}{2}x^3 - 12\frac{3}{4}x^4 - 29\frac{5}{8}x^5 - 59\frac{2}{5}x^6, \&c. \\
\text{- - - - - - - - - - -} + \frac{9}{2}x^3 + 6x^4 + \frac{75}{8}x^5 + \frac{273}{20}x^6, \&c. \\
\text{- - - - - - - - - - - - - - -} + 9x^4 + 12x^5 + \frac{75}{4}x^6, \&c. \\
\text{- - - - - - - - - - - - - - -} - \frac{9}{4}x^4 - 6x^5 - \frac{107}{8}x^6, \&c. \\
\text{- - - - - - - - - - - - - - - - - - -} + 12x^5 + 16x^6, \&c. \\
\text{- - - - - - - - - - - - - - - - - - -} + \frac{9}{4}x^5 + 6x^6, \&c. \\
\text{- -} + 15x^6, \&c. \\
\text{- -} + \frac{27}{8}x^6, \&c.
\end{array}$$

$$y = -\frac{3}{2}x^2 - 2x^3 - \frac{25}{8}x^4 - \frac{91}{20}x^5 - \frac{111}{16}x^6 - \frac{367}{35}x^7, \&c.$$

Make $\dot{y} = -3x$, &c. then $y = -\frac{3}{2}x^2$, &c. Then $\dot{y} = * - 6x^2$, &c. and $y = * - 2x^3$, &c. Then $-3xy = +\frac{9}{2}x^3$, &c. and therefore $\dot{y} = * * - \frac{9}{2}x^3 - 8x^3$, &c. $= * * - \frac{25}{2}x^3$, &c. and $y = * * - \frac{25}{8}x^4$, &c. And ſo of the reſt.

The Author here takes notice, that as the value of $\dot{y}$ is negative, and therefore contrary to that of $\dot{x}$, it ſhews that as x increaſes, y muſt decreaſe, and on the contrary. For a negative Velocity is a Velocity backwarks, or whoſe direction is contrary to that which was ſuppos'd to be an affirmative Velocity. This Remark muſt take place hereafter, as often as there is occaſion for it.

38. In this Example the Author puts x to repreſent the Relate Quantity, or the Root to be extracted, and y to repreſent the Correlate. But to prevent the confuſion of Ideas, we ſhall here change x into y, and y into x, ſo that y ſhall denote the Relate, and x the Correlate Quantity, as uſual. Let the given Equation therefore be $\frac{\dot{y}}{\dot{x}} = \frac{1}{2}x - 4x^2 + 2xy^{\frac{1}{2}} - \frac{4}{3}y^2 + 7x^{\frac{5}{2}} + 2x^3$, whoſe Root y is to be extracted. Theſe Terms being diſpoſed in a Table, will ſtand thus: And the Reſolution will be as follows, taking $-\dot{y}$ and $+\frac{1}{2}x$ for the two external Terms.

$$\left.\begin{array}{c|ccccc}
 & y^2 & y^{\frac{3}{2}} & y & y^{\frac{1}{2}} & y^0 \\
\hline
x^3 & * & * & * & * & +2x^3 \\
x^{\frac{5}{2}} & * & * & * & * & +7x^{\frac{5}{2}} \\
x^2 & * & * & * & * & -4x^2 \\
x^{\frac{3}{2}} & * & * & * & * & * \\
x^1 & * & * & * & +2xy^{\frac{1}{2}} & +\frac{1}{2}x \\
x^{\frac{1}{2}} & * & * & * & * & * \\
x^0 & -\frac{4}{3}y^2 & * & * & * & * \\
x^{-\frac{1}{2}} & * & * & * & * & * \\
x^{-1} & * & * & -\dot{y} & * & *
\end{array}\right\} = 0.$$

$$\begin{array}{r}
\left.\begin{array}{r} \dot{y} \\ \\ -2xy^{\frac{1}{2}} \\ +\frac{4}{3}y^2 \end{array}\right\}
\begin{array}{l}
= \frac{1}{2}x \quad * \quad -4x^2 + 7x^{\frac{5}{2}} + 2x^3 \\
\qquad\quad + x^2 \qquad\quad -2x^3 + 4x^{\frac{7}{2}} - \frac{41}{20}x^4, \text{ \&c.} \\
- - - - - x^2 \quad * \quad +2x^3 - 4x^{\frac{7}{2}} + 2x^4, \text{ \&c.} \\
- - - - - - - - - - - - - - + \frac{1}{20}x^4, \text{ \&c.}
\end{array} \\
y = \frac{1}{4}x^2 \quad * \quad -x^3 + 2x^{\frac{7}{2}} \quad * \quad + \frac{8}{9}x^{\frac{9}{2}} - \frac{41}{100}x^5, \text{ \&c.}
\end{array}$$

Make $\dot{y} = \frac{1}{2}x$, &c. then $y = \frac{1}{4}x^2$, &c. Now becauſe it is $\dot{y} = * \; 0$, &c. it will be alſo $y = * \; 0$, &c. And whereas it is $y^{\frac{1}{2}} = \frac{1}{2}x$, &c. it will be $-2xy^{\frac{1}{2}} = -x^2$, &c. and therefore $\dot{y} = * \; * + x^2 - 4x^2$, &c. $= * \; * - 3x^2$, &c. then $y = * \; * - x^3$, &c. Now becauſe it is $y = * + 0$, &c. it will be alſo $y^{\frac{1}{2}} = * + 0$, &c. and $-2xy^{\frac{1}{2}} = * + 0$, &c. and conſequently $\dot{y} = * \; * \; * + 7x^{\frac{5}{2}}$, &c. and therefore $y = * \; * \; * + 2x^{\frac{7}{2}}$, &c. And ſo on.

There are two other caſes of external Terms, which will ſupply us with two other Series for the Root y, but they will run too much into Surds. This may be ſufficient to ſhew the univerſality of the Method, and how we are to proceed in like caſes.

39. The Author ſhews here, that the ſame Fluxional Equation may often afford a great variety of Series for the Root, according as we ſhall introduce any conſtant quantity at pleaſure. Thus the Equation of Art. 34. or $\dot{y} = 1 - 3x + y + x^2 + xy$, may be reſolved after the following general manner:

$$\left.\begin{array}{r} \dot{y} \\ \\ \\ -y \\ \\ -xy \\ \\ \end{array}\right\} \begin{array}{l} =1-3x+x^2 \\ +a+x+2ax^2+\frac{5}{3}ax^3, \&c. \\ \quad +2ax \qquad -\frac{2}{3}x^3 \\ -a-x+x^2-\frac{1}{3}x^3, \&c. \\ \quad -ax-ax^2-\frac{2}{3}ax^3 \\ \quad -ax-x^2-ax^3, \&c. \\ \qquad -ax^2+x^3 \end{array} \qquad \begin{array}{l} y=a+x-x^2+\frac{1}{3}x^3-\frac{1}{6}x^4, \&c. \\ \quad +ax+ax^2+\frac{2}{3}ax^3+\frac{5}{12}ax^4 \end{array}$$

Here inſtead of making $\dot{y}=1$, &c. we may make $\dot{y}=0$, &c. and therefore $y=a$, &c. becauſe then $\dot{y}=0$, &c. then $-y=-a$, &c. and conſequently $\dot{y}=*+a+1$, &c. and therefore $y=*+ax+x$, &c. Then $-y=*-ax-x$, &c. and $-xy=-ax$, &c. and therefore $\dot{y}=**+2ax+x-3x$, &c. $=**+2ax-2x$, &c. and then $y=**+ax^2-x^2$, &c. Therefore $-y=**-ax^2+x^2$, &c. and $-xy=*-ax^2-x^2$, &c. and conſequently $\dot{y}=***+2ax^2+x^2$, &c. and $y=***+\frac{2}{3}ax^3+\frac{1}{3}x^3$, &c. &c. Here if we make $a=0$, we ſhall have the ſame value of y as was extracted before. And by whatever Number a is interpreted, ſo many different Series we ſhall obtain for y.

40. The Author here enumerates three caſes, when an arbitrary Number ſhould be aſſumed, if it can be done, for the firſt Term of the Root. Firſt, when in the given Equation the Root is affected with a Fractional Dimenſion, or when ſome Root of it is to be extracted; for then it is convenient to have Unity for the firſt Term, or ſome other Number whoſe Root may be extracted without a Surd, if ſuch Number does not offer itſelf of its own accord. As in the fourth Example 'tis $x=\frac{1}{4}y^2$, &c. and therefore we may eaſily have $x^{\frac{1}{2}}=\frac{1}{2}y$, &c. Secondly, it muſt be done, when by reaſon of the ſquare-root of a negative Quantity, we ſhould otherwiſe fall upon impoſſible Numbers. Laſtly, we muſt aſſume ſuch a Number, when otherwiſe there would be no initial Quantity, from whence to begin the computation of the Root; that is, when the Relate Quantity, or its Fluxion, affects all the Terms of the Equation.

41, 42, 43. The Author's Compendiums of Extraction are very curious, and ſhew the univerſality of his Method. As his ſeveral Proceſſes want no explanation, I ſhall proceed to reſolve his Examples by the foregoing general Method. As if the given Equation were $\dot{y}=\frac{1}{y}-x^2$, or $\dot{y}-y^{-1}=-x^2$, the Reſolution might be thus:

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$$\begin{array}{r|llllll}
\dot{y} & = 0 & * & * & - & x^2 & -\frac{5}{2}a^{-7}x^3, \text{ \&c.} \\
 & & +a^{-1} & -a^{-3}x & +\frac{3}{2}a^{-5}x^2 & & +\frac{1}{3}a^{-2}x^3 \\
-y^{-1} & --- & -a^{-1} & +a^{-3}x & -\frac{3}{2}a^{-5}x^2 & & +\frac{5}{2}a^{-7}x^3, \text{ \&c.} \\
 & & & & & & -\frac{1}{3}a^{-2}x^3 \\
\end{array}$$

$$y = a + \frac{x}{a} - \frac{x^2}{2a^3} + \frac{x^3}{2a^5} - \frac{5x^4}{8a^7}, \text{ \&c.}$$
$$-\frac{1}{3}x^3 + \frac{x^4}{12a^2}$$

Make $\dot{y} = 0$, &c. then aſſuming any conſtant quantity a, it may be $y = a$, &c. Then by Diviſion $-y^{-1} = -a^{-1}$, &c. and therefore $\dot{y} = * + a^{-1}$, &c. and conſequently $y = * + a^{-1}x$, &c. Then by Diviſion $-y^{-1} = * + a^{-3}x$, &c. and therefore $\dot{y} = * * - a^{-3}x$, &c. and conſequently $y = * * - \frac{1}{2}a^{-3}x^2$, &c. Then again by Diviſion $-y^{-1} = * * - \frac{3}{2}a^{-5}x^2$, &c. and therefore $\dot{y} = * * * + \frac{3}{2}a^{-5}x^2 - x^2$, &c. and conſequently $y = * * * \frac{1}{2}a^{-5}x^3 - \frac{1}{3}x^3$, &c. And ſo of the reſt. Here if we make $a = 1$, we ſhall have $y = 1 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{13}{24}x^4$, &c.

Or the ſame Equation may be thus reſolved:

$$\begin{array}{r|l}
-y^{-1} & = -x^2 + 2x^{-3} + 14x^{-8} + 216x^{-13}, \text{ \&c.} \\
+\dot{y} & ------- 2x^{-3} - 14x^{-8} - 216x^{-13}, \text{ \&c.} \\
\end{array}$$
$$y = x^{-2} + 2x^{-7} + 18x^{-12} + 280x^{-17}, \text{ \&c.}$$

Make $-y^{-1} = -x^2$, &c. or $y = x^{-2}$, &c. Then $\dot{y} = -2x^{-3}$, &c. and therefore $-y^{-1} = * + 2x^{-3}$, &c. and conſequently by Diviſion $y = * + 2x^{-7}$, &c. Then $\dot{y} = * - 14x^{-8}$, &c. and therefore $-y^{-1} = * * + 14x^{-8}$, &c. and by Diviſion $y = * * + 18x^{-12}$, &c. Then $\dot{y} = * * - 216x^{-13}$, &c. and therefore $-y^{-1} = * * * + 216x^{-13}$, &c. and by Diviſion $y = * * * + 280x^{-17}$, &c. And ſo on.

Another aſcending Series may be had from this Equation, *viz.* $y = \sqrt{2x} - \frac{2}{7}x^3 + \frac{x^{\frac{11}{2}}}{147\sqrt{2}} + \frac{10x^8}{17493}$, &c. by multipying it by y, and then making 1 the firſt Term of $y\dot{y}$.

44. The Equation $\dot{y} = 3 + 2y - x^{-1}y^2$ may be thus reſolved:

$$\begin{array}{r|l}
\dot{y} & = 3 - 3x + 6x^2, \text{ \&c.} \\
-2y & --- - 6x + 3x^2, \text{ \&c.} \\
+x^{-1}y^2 & --- + 9x - 9x^2, \text{ \&c.} \\
\end{array}$$
$$y = 3x - \frac{3}{2}x^2 + 2x^3, \text{ \&c.}$$

$$\begin{array}{c|ccc}
 & y^2 & y^1 & y^0 \\
\hline
x^0 & * & +2y & +3 \\
x^{-1} & -x^{-1}y^2 & -\dot{y} & * \\
\end{array}\Bigg\} = 0.$$

Make $\dot{y} = 3$, &c. then $y = 3x$, &c. Therefore $-2y = -6x$, &c. and $x^{-1}y^2 = 9x$, &c. and conſequently $\dot{y} = * - 3x$, &c. Therefore $y = * - \frac{3}{2}x^2$, &c. Then $-2y = * + 3x^2$, &c. and $x^{-1}y^2 = * - 9x^2$, &c. Therefore $\dot{y} = * * + 6x^2$, &c. and $y = * * + 2x^3$, &c. &c.

Or the Reſolution may be perform'd after theſe two following manners:

$$\left.\begin{array}{l} -2y \\ +\dot{y} \\ +x^{-1}y^2 \end{array}\right\} \begin{array}{l} = 3 - \frac{9}{4}x^{-1} + \frac{9}{2}x^{-2}, \&c. \\ \text{- - - -} \;*\; - \frac{9}{8}x^{-2}, \&c. \\ \text{- -} \; + \frac{9}{4}x^{-1} - \frac{27}{8}x^{-2}, \&c. \end{array} \qquad \left.\begin{array}{l} yx^{-1} \\ +\dot{y}y^{-1} \\ -3y^{-1} \end{array}\right\} \begin{array}{l} = 2 + \frac{1}{2}x^{-1} - \frac{3}{8}x^{-2}, \&c. \\ \text{- -} \; + x^{-1} \;\; * \\ \quad - \frac{3}{2}x^{-1} + \frac{3}{8}x^{-2}, \&c. \end{array}$$

$$y = -\tfrac{3}{2} + \tfrac{9}{8}x^{-1} - \tfrac{9}{4}x^{-2}, \&c. \qquad y = 2x + \tfrac{1}{2} - \tfrac{3}{8}x^{-1}, \&c.$$

Make $-2y = 3$, &c. or $y = -\frac{3}{2}$, &c. then $\dot{y} = 0$, &c. and $x^{-1}y^2 = +\frac{9}{4}x^{-1}$, &c. Therefore $-2y = * - \frac{9}{4}x^{-1}$, &c. or $y = * + \frac{9}{8}x^{-1}$, &c. and $\dot{y} = * - \frac{9}{8}x^{-2}$, &c. and by ſquaring $x^{-1}y^2 = * - \frac{27}{8}x^{-2}$, &c. and therefore $-2y = * * + \frac{9}{2}x^{-2}$, &c. and $y = * * - \frac{9}{4}x^{-2}$, &c. And ſo on.

Again, divide the whole Equation by y, and make $x^{-1}y = 2$, &c. then $y = 2x$, &c. And becauſe $\dot{y} = 2$, &c. and $y^{-1} = \frac{1}{2}x^{-1}$, &c. 'tis $\dot{y}y^{-1} = x^{-1}$, &c. and $-3y^{-1} = -\frac{3}{2}x^{-1}$, &c. therefore $yx^{-1} = * + \frac{1}{2}x^{-1}$, &c. and $y = * + \frac{1}{2}$, &c. Then becauſe $\dot{y}y^{-1} = * + 0$, &c. and $-3y^{-1} = * + \frac{3}{8}x^{-2}$, &c. 'tis $yx^{-1} = * * - \frac{3}{8}x^{-2}$, &c. and $y = * * - \frac{3}{8}x^{-1}$, &c. &c.

45, 46. If the propoſed Equation be $\dot{y} = -y + x^{-1} - x^{-2}$, its Solution may be thus:

$$\left.\begin{array}{l} \dot{y} \\ \\ +y \end{array}\right\} \begin{array}{r} = -x^{-2} + x^{-1} \\ -x^{-1} \\ \text{- - - - -} \;\; +x^{-1} \end{array} \qquad \left.\begin{array}{l} y \\ \\ +\dot{y} \end{array}\right\} \begin{array}{r} = x^{-1} - x^{-2} \\ +x^{-2} \\ \text{- - -} \;\; -x^{-2} \end{array} \qquad \begin{array}{c|cc} & y^1 & y^0 \\ \hline x^0 & -y & * \\ x^{-1} & -\dot{y} & +x^{-1} \\ x^{-2} & * & -x^{-2} \end{array} \Bigg\} = 0.$$

$$y = x^{-1} \qquad\qquad y = x^{-1}$$

Make $\dot{y} = -x^{-2}$, &c. then $y = x^{-1}$, &c. Conſequently $\dot{y} = * \; 0$, &c. and therefore $y = * \; 0$, &c. that is, $y = x^{-1}$.

Again, make $y = x^{-1}$, &c. then $\dot{y} = -x^{-2}$, &c. and conſequently $y = * + 0$, &c. that is, $y = x^{-1}$.

That this ſhould be ſo, may appear by the direct Method. For if $y = x^{-1}$, 'tis $\dot{y} = -\dot{x}x^{-2}$; alſo $y\dot{x} = \dot{x}x^{-1}$. Then adding theſe two Equations together, 'tis $y\dot{x} + \dot{y} = \dot{x}x^{-1} - \dot{x}x^{-2}$, or $\dot{y} = -y + x^{-1} - x^{-2}$. Thus may we form as many Fluxional Equations as we pleaſe, of which the Fluents may be expreſs'd in finite Terms; but to return to theſe again may ſometimes require particular Expedients. Thus if we aſſume the Equation $y = 2x - \frac{4}{3}x^2 + \frac{1}{5}x^3$, taking the Fluxions, and putting $\dot{x} = 1$, we ſhall have $\dot{y} = 2 - \frac{8}{3}x + \frac{3}{5}x^2$, as alſo $\frac{y}{2x} = 1 - \frac{2}{3}x + \frac{1}{10}x^2$. Subtract this laſt from the foregoing Equation, and we ſhall have $\dot{y} - \frac{y}{2x} = 1 - 2x + \frac{1}{2}x^2$, the Solution of which here follows.

47. Let the propos'd Equation be $\dot{y} = \frac{y}{2x} + 1 - 2x + \frac{1}{2}x^2$, of which the Solution may be thus:

$$\begin{array}{r} \dot{y} \\ \\ -\frac{y}{2x} \end{array} \left\} \begin{array}{lll} = 1 & -2x & +\frac{1}{2}x^2 \\ +e & +fx & +gx^2 \\ -e & -fx & -gx^2 \end{array} \right. \qquad \begin{array}{r} \dot{y} \\ \\ -\frac{y}{2x} \end{array} \left\} \begin{array}{lll} = \frac{1}{2}x^2 & -2x & +1 \\ +ex^2 & +fx & +g \\ -ex^2 & -fx & -g \end{array} \right.$$

$$y = 2ex + 2fx^2 + 2gx^3 \qquad\qquad y = 2ex^3 + 2fx^2 + 2g$$

$$y = 2x - \frac{4}{3}x^2 + \frac{1}{5}x^3 \qquad\qquad y = \frac{1}{3}x^3 - \frac{4}{3}x^2 + 2x$$

$$\left.\begin{array}{c|cc} & y^1 & y^0 \\ \hline x^2 & * & +\frac{1}{2}x^2 \\ x^1 & * & -2x \\ x^0 & * & +1 \\ x^{-1} & \left.\begin{array}{c} \dot{y} \\ +\frac{1}{2}yx^{-1} \end{array}\right\} & * \end{array}\right\} = 0.$$

By tabulating the Terms of this Equation, as uſual, it may be obſerved, that one of the external Terms $-\dot{y} + \frac{1}{2}yx^{-1}$ is a double Term, to which the other external Term 1 belongs in common. Therefore to ſeparate theſe, aſſume $y = 2ex$, &c. then $-\frac{y}{2x} = -e$, &c. and conſequently $\dot{y} = 1 + e$, &c. and therefore $y = x + ex$, &c. That is, becauſe $2ex = x + ex$, or $2e = 1 + e$, 'tis $e = 1$, or $y = 2x$, &c. So if we make $y = * + 2fx^2$, &c. then $-\frac{y}{2x} = * - fx$, &c. therefore $\dot{y} = * + fx - 2x$, &c. and $y = * + \frac{1}{2}fx^2 - x^2$, &c. that is, $2f = \frac{1}{2}f - 1$, or $f = -\frac{2}{3}$. So that $y = * - \frac{4}{3}x^2$, &c. So if we make $y = * * + 2gx^3$, &c. then $-\frac{y}{2x} = * * - gx^2$, &c. and therefore $\dot{y} = * * + gx^2 + \frac{1}{2}x^2$, &c. and $y = * * + \frac{1}{3}gx^3 + \frac{1}{6}x^3$, &c. or $2g = \frac{1}{3}g + \frac{1}{6}$, or $g = \frac{1}{10}$, ſo that $y = * * \frac{1}{5}x^3$, &c. So if we make $y = * * * 2hx^4$, &c. then $-\frac{y}{2x} = * * * - hx^3$; &c. and therefore $\dot{y} = * * * + hx^3$, &c. and $y = * * * + \frac{1}{4}hx^4$, &c. But becauſe here $2h = \frac{1}{4}h$, this Equation would be abſurd except $h = 0$. And ſo all the ſubſequent Terms will vaniſh *in infinitum*, and this will be the exact value of y. And the ſame may be done from the other caſe of external Terms, as will appear from the Paradigm.

48. Nothing can be added to illuſtrate this Inveſtigation, unleſs we would demonſtrate it ſynthetically. Becauſe $y = ex^{\frac{3}{4}}$, as is here

found, therefore $\dot{y} = \frac{3}{4}ex^{\frac{3}{4}-1}$, or $\dot{y} = \frac{3ex^{\frac{3}{4}}}{4x}$. Here inſtead of $ex^{\frac{3}{4}}$ ſubſtitute y, and we ſhall have $\dot{y} = \frac{3y}{4x}$, as given at firſt.

49, 50. The given Equation $\dot{y} = yx^{-2} + x^{-2} + 3 + 2x - 4x^{-1}$ may be thus reſolved after a general manner.

$$\begin{array}{r}\left.\begin{array}{r}\dot{y}\\ \\ -x^{-2}y\\ \\ \end{array}\right\}\end{array}\begin{array}{llllllll}= 2x & +3 & -4x^{-1} & +x^{-2} & -x^{-3} & +\frac{1}{2}x^{-4}, \&c. & & \\ & +1 & +4x^{-1} & +ax^{-2} & -ax^{-3} & +\frac{1}{2}ax^{-4} & & \\ ---- & -1 & -4x^{-1} & -ax^{-2} & +x^{-3} & -\frac{1}{2}x^{-4}, \&c. & & \\ & & & & +ax^{-3} & -\frac{1}{2}ax^{-4} & & \\ y = x^2 & +4x & +a & -x^{-1} & +\frac{1}{2}x^{-2} & -\frac{1}{6}x^{-3}, \&c. & & \\ & & & -ax^{-1} & +\frac{1}{2}ax^{-2} & -\frac{1}{6}ax^{-3}. & & \end{array}$$

Make $\dot{y} = 2x$, &c. then $y = x^2$, &c. Therefore $-x^{-2}y = -1$, &c. conſequently $\dot{y} = * + 1 + 3$, &c. $= * 4$, &c. and therefore $y = * + 4x$, &c. Then $-x^{-2}y = * - 4x^{-1}$, &c. and conſequently $\dot{y} = * * + 0$, &c. and therefore aſſuming any conſtant quantity a, it may be $y = * * + a$, &c. Then $-x^{-2}y = * * - ax^{-2}$, &c. and therefore $\dot{y} = * * * + ax^{-2} + x^{-2}$, &c. and $y = * * * - ax^{-1} - x^{-1}$, &c. And ſo on. Here if we make $a = 0$, 'tis $y = x^2 + 4x * - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3}$, &c.

51, 52. The Equation of this Example is $\dot{y} = 3xy^{\frac{2}{3}} + y$, which we ſhall reſolve by our uſual Method, without any other preparation than dividing the whole by $y^{\frac{2}{3}}$, that one of the Terms may be clear'd from the Relate Quantity; which will reduce it $\dot{y}y^{-\frac{2}{3}} - y^{\frac{1}{3}} = 3x$, of which the Reſolution may be thus:

$$\left.\begin{array}{r}\dot{y}y^{-\frac{2}{3}}\\ -y^{\frac{1}{3}}\end{array}\right\}\begin{array}{lllll}= 3x & +\frac{1}{2}x^2 & +\frac{1}{18}x^3 & +\frac{1}{216}x^4 & +\frac{1}{3240}x^5, \&c.\\ ---- & -\frac{1}{2}x^2 & -\frac{1}{18}x^3 & -\frac{1}{216}x^4 & -\frac{1}{3240}x^5, \&c.\\ & y = \frac{1}{8}x^6 & +\frac{1}{24}x^7 & +\frac{1}{288}x^8, \&c. & \end{array}$$

Make $\dot{y}y^{-\frac{2}{3}} = 3x$, &c. or taking the Fluents, $3y^{\frac{1}{3}} = \frac{3}{2}x^2$, &c. or $y^{\frac{1}{3}} = \frac{1}{2}x^2$, &c. or $y = \frac{1}{8}x^6$, &c. And becauſe $-y^{\frac{1}{3}} = -\frac{1}{2}x^2$, &c. it will be $\dot{y}y^{-\frac{2}{3}} = * + \frac{1}{2}x^2$, &c. and therefore $3y^{\frac{1}{3}} = * + \frac{1}{6}x^3$, &c. and $y^{\frac{1}{3}} = * + \frac{1}{18}x^3$, &c. and by cubing $y = * + \frac{1}{24}x^7$, &c. Then becauſe $-y^8 = * - \frac{1}{18}x^3$, &c. 'tis $\dot{y}y^{-\frac{2}{3}} = * * + \frac{1}{18}x^3$, &c. and therefore $3y^{\frac{1}{3}} = * * + \frac{1}{72}x^4$, &c. and $y^{\frac{1}{3}} = * * + \frac{1}{216}x^4$, &c. and by cubing $y = * * + \frac{1}{288}x^8$, &c. And ſo on.

53.

53. Laſtly, in the Equation $\dot{y} = 2y^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}}$, or $\dot{y}y^{-\frac{1}{2}} = 2\dot{x} + \dot{x}x^{\frac{1}{2}}$, aſſuming c for a conſtant quantity, whoſe Fluxion therefore is o, and taking the Fluents, it will be $2y^{\frac{1}{2}} = 2c + 2x + \frac{2}{3}x^{\frac{3}{2}}$, or $y^{\frac{1}{2}} = c + x + \frac{1}{3}x^{\frac{3}{2}}$. Then by ſquaring, $y = c^2 + 2cx + x^2 + \frac{2}{3}cx^{\frac{3}{2}} + \frac{2}{3}x^{\frac{5}{2}} + \frac{1}{9}x^3$. Here the Root y may receive as many different values, while x remains the ſame, as c can be interpreted different ways. Make $c = 0$, then $y = x^2 + \frac{2}{3}x^{\frac{5}{2}} + \frac{1}{9}x^3$.

The Author is pleas'd here to make an Excuſe for his being ſo minute and particular, in diſcuſſing matters which, as he ſays, will but ſeldom come into practice; but I think any Apology of this kind is needleſs, and we cannot be too minute, when the perfection of a Method is concern'd. We are rather much obliged to him for giving us his whole Method, for applying it to all the caſes that may happen, and for obviating every difficulty that may ariſe. The uſe of theſe Extractions is certainly very extenſive; for there are no Problems in the inverſe Method of Fluxions, and eſpecially ſuch as are to be anſwer'd by infinite Series, but what may be reduced to ſuch Fluxional Equations, and may therefore receive their Solutions from hence. But this will appear more fully hereafter.

SECT. IV. *Solution of the third Caſe of Equations, with ſome neceſſary Demonſtrations.*

54. FOR the more methodical Solution of what our Author calls *a moſt troubleſome and difficult Problem*, (and ſurely the Inverſe Method of Fluxions, in its full extent, deſerves to be call'd ſuch a Problem,) he has before diſtributed it into three Caſes. The firſt Caſe, in which two Fluxions and only one flowing Quantity occur in the given Equation, he has diſpatch'd without much difficulty, by the aſſiſtance of his Method of infinite Series. The ſecond Caſe, in which two flowing Quantities and their Fluxions are any how involved in the given Equation, even with the ſame aſſiſtance is ſtill an operoſe Problem, but yet is diſcuſs'd in all its varieties, by a ſufficient number of appoſite Examples. The third Caſe, in which occur more than two Fluxions with their Fluents, is here very artfully managed, and all the difficulties of it are reduced to the other two Caſes. For if the Equation involves (for inſtance) three Fluxions, with ſome or all of their Fluents, another Equation ought to be given by the Queſtion, in order to a full Determination,

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termination, as has been already argued in another place; or if not, the Queſtion is left indetermined, and then another Equation may be aſſumed *ad libitum*, ſuch as will afford a proper Solution to the Queſtion. And the reſt of the work will only require the two former Caſes, with ſome common Algebraic Reductions, as we ſhall ſee in the Author's Example.

55. Now to conſider the Author's Example, belonging to this third Caſe of finding Fluents from their Fluxions given, or when there are more than two variable Quantities, and their Fluxions, either expreſs'd or underſtood in the given Equation. This Example is $2\dot{x} - \dot{z} + \dot{y}x = 0$, in which becauſe there are three Fluxions $\dot{x}$, $\dot{y}$, and $\dot{z}$, (and therefore virtually three Fluents x, y, and z,) and but one Equation given; I may aſſume (for inſtance) $x = y$, whence $\dot{x} = \dot{y}$, and by ſubſtitution $2\dot{y} - \dot{z} + \dot{y}y = 0$, and therefore $2y - z + \frac{1}{2}y^2 = 0$. Now as here are only two Equations $x - y = 0$ and $2y - z + \frac{1}{2}y^2 = 0$, the Quantities x, y, and z are ſtill variable Quantities, and ſuſceptible of infinite values, as they ought to be. Indeed a third Equation may be had, as $2x - z + \frac{1}{2}x^2 = 0$; but as this is only derived from the other two, it brings no new limitation with it, but leaves the quantities ſtill flowing and indeterminate quantities. Thus if I ſhould aſſume $2y = a + z$ for the ſecond Equation, then $2\dot{y} = \dot{z}$, and by ſubſtitution $2\dot{x} - 2\dot{y} + \dot{y}x = 0$, or $\dot{y} = \frac{2\dot{x}}{2 - x} = \dot{x} + \frac{1}{2}x\dot{x} + \frac{1}{4}x^2\dot{x}$, &c. and therefore $y = x + \frac{1}{4}x^2 + \frac{1}{12}x^3$, &c. which two Equations are a compleat Determination. Again, if we aſſume with the Author $x = y^2$, and thence $\dot{x} = 2y\dot{y}$, we ſhall have by ſubſtitution $4y\dot{y} - \dot{z} + \dot{y}y^2 = 0$, and thence $2y^2 - z + \frac{1}{3}y^3 = 0$, which two Equations are a ſufficient Determination. We may indeed have a third, $2x - z + \frac{1}{3}x^{\frac{3}{2}} = 0$; but as this is included in the other two, and introduces no new limitation, the quantities will ſtill remain fluent. And thus an infinite variety of ſecond Equations may be aſſumed, tho' it is always convenient, that the aſſumed Equation ſhould be as ſimple as may be. Yet ſome caution muſt be uſed in the choice, that it may not introduce ſuch a limitation, as ſhall be inconſiſtent with the Solution. Thus if I ſhould aſſume $2x - z = 0$ for the ſecond Equation, I ſhould have $2\dot{x} - \dot{z} = 0$ to be ſubſtituted, which would make $\dot{y}x = 0$, and therefore would afford no Solution of the Equation.

'Tis eaſy to extend this reaſoning to Equations, that involve four or more Fluxions, and their flowing Quantities; but it would be needleſs here to multiply Examples. And thus our Author has compleatly ſolved this Caſe alſo, which at firſt view might appear formidable enough, by reducing all its difficulties to the two former Caſes.

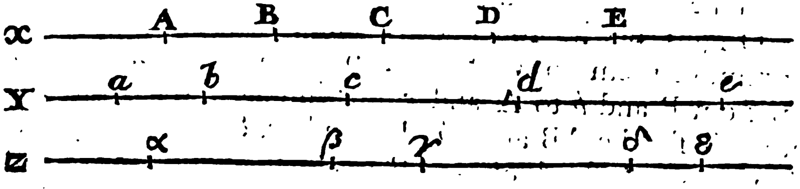
56, 57. The Author's way of demonſtrating the Inverſe Method of Fluxions is ſhort, but ſatisfactory enough. We have argued elſewhere, that from the Fluents given to find the Fluxions, is a direct and ſynthetical Operation; and on the contrary, from the Fluxions given to find the Fluents, is indirect and analytical. And in the order of nature Syntheſis ſhould always precede Analyſis, or Compoſition ſhould go before Reſolution. But the Terms Syntheſis and Analyſis are often uſed in a vague ſenſe, and taken only relatively, as in this place. For the direct Method of Fluxions being already demonſtrated ſynthetically, the Author declines (for the reaſons he gives) to demonſtrate the Inverſe Method ſynthetically alſo, that is, primarily, and independently of the direct Method. He contents himſelf to prove it analytically, that is, by ſuppoſing the direct Method, as ſufficiently demonſtrated already, and ſhewing the neceſſary connexion between this and the inverſe Method. And this will always be a full proof of the truth of the concluſions, as Multiplication is a good proof of Diviſion. Thus in the firſt Example we found, that if the given Equation is $y + xy - \dot{y} = 3x - x^2 - 1$, we ſhall have the Root $y = x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6$, &c. To prove the truth of which concluſion, we may hence find, by the direct Method, $\dot{y} = 1 - 2x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{2}{15}x^5$, &c. and then ſubſtitute theſe two Series in the given Equation, as follows:

$$\begin{array}{rrrrrrrrl}
y \text{ - - - - - -} & & +x & -x^2 & +\frac{1}{3}x^3 & -\frac{1}{6}x^4 & +\frac{1}{30}x^5 & -\frac{1}{45}x^6, & \text{\&c.} \\
+xy \text{ - - - - - - - - -} & & & +x^2 & -x^3 & +\frac{1}{3}x^4 & -\frac{1}{6}x^5 & +\frac{1}{30}x^6, & \text{\&c.} \\
-\dot{y} \text{ - -} & -1 & +2x & -x^2 & +\frac{2}{3}x^3 & -\frac{1}{6}x^4 & +\frac{2}{15}x^5 & -\frac{1}{90}x^6, & \text{\&c.} \\
\hline
= & -1 & +3x & -x^2 & * & * & * & * &
\end{array}$$

Now by collecting theſe Series, we ſhall find the reſult to produce the given Equation, and therefore the preceding Operation will be ſufficiently proved.

58. In this and the ſubſequent paragraphs, our Author comes to open and explain ſome of the chief Myſteries of Fluxions and Fluents, and to give us a Key for the clearer apprehenſion of their nature and properties. Therefore for the Learners better inſtruction, I ſhall not think much to inquire ſomething more circumſtantially into this matter. In order to which let us conceive any number of right Lines, AE, *ae*, *ae*, &c. indefinitely extended both ways, along which a Body, or a deſcribing Point, may be ſuppoſed to move in each

Line, from the left-hand towards the right, according to any Law or Rate of Acceleration or Retardation whatever. Now the Motion of every one of theſe Points, at all times, is to be eſtimated by its diſtance from ſome fixt point in the ſame Line; and any ſuch Points may be choſen for this purpoſe, in each Line, ſuppoſe B, b, β, in which all the Bodies have been, are, or will be, in the ſame Moment of Time, from whence to compute their contemporaneous Augments, Differences, or flowing Quantities. Theſe Fluents may be conceived as negative before the Body arrives at that point, as nothing when in it, and as affirmative when they are got beyond it. In the firſt Line AE, whoſe Fluent we denominate by x, we may ſuppoſe the Body to move uniformly, or with any equable Velocity; then may the Fluent x, or the Line which is continually deſcribed,



repreſent Time, or ſtand for the Correlate Quantity, to which the ſeveral Relate-Quantities are to be conſtantly refer'd and compared. For in the ſecond Line ae, whoſe Fluent we call y, if we ſuppoſe the Body to move with a Motion continually accelerated or retarded, according to any conſtant Rate or Law, (which Law is expreſs'd by any Equation compos'd of x and y and known quantities;) then will there always be contemporaneous parts or augments, deſcribed in the two Lines, which parts will make the whole Fluents to be contemporaneous alſo, and accommodate themſelves to the Equation in all its Circumſtances. So that whatever value is aſſumed for the Correlate x, the correſponding or contemporaneous value of the Relate y may be known from the Equation, and *vice verſâ*. Or from the Time being given, here repreſented by x, the Space repreſented by y may always be known. The Origin (as we may call it) of the Fluent x is mark'd by the point B, and the Origin of the Fluent y by the point b. If the Bodies at the ſame time are found in A and a, then will the contemporaneous Fluents be $-BA$ and $-ba$. If at the ſame time, as was ſuppoſed, they are found in their reſpective Origins B and b, then will each Fluent be nothing. If at the ſame time they are found in C and c, then will their Fluents be $+BC$ and $+bc$. And the like of all other points, in which the moving Bodies either have been, or ſhall be found, at the ſame time.

As to the Origins of theſe Fluents, or the points from whence we begin to compute them, (for tho' they muſt be conceived to be variable and indetermined in reſpect of one of their Limits, where the deſcribing points are at preſent, yet they are fixt and determined as to their other Limit, which is their Origin,) tho' before we appointed the Origin of each Fluent to be in B and *b*, yet it is not of abſolute neceſſity that they ſhould begin together, or at the ſame Moment of Time. All that is neceſſary is this, that the Motions may continue as before, or that they may obſerve the ſame rate of flowing, and have the ſame contemporaneous Increments or Decrements, which will not be at all affected by changing the beginnings of the Fluents. The Origins of the Fluents are intirely arbitrary things, and we may remove them to what other points we pleaſe. If we remove them from B and *b* to A and *c*, for inſtance, the contemporaneous Lines will ſtill be AB and *ab*, BC and *bc*, &c. tho' they will change their names. Inſtead of $-$ AB we ſhall have o, inſtead of B or o we ſhall have $+$ AB, inſtead of $+$ BC we ſhall have $+$ AC; &c. So inſtead of $-ab$ we ſhall have $-ac+bc$, inſtead of *b* or o we ſhall have $-bc$, inſtead of $+bd$ we ſhall have $+bc+cd$, &c. That is, in the Equation which determines the general Law of flowing or increaſing, we may always increaſe or diminiſh x, or y, or both, by any given quantity, as occaſion may require, and yet the Equation that ariſes will ſtill expreſs the rate of flowing; which is all that is neceſſary here. Of the uſe and conveniency of which Reduction we have ſeen ſeveral inſtances before.

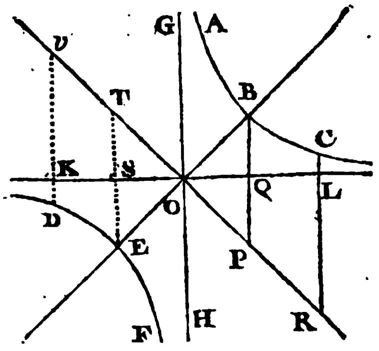
If there be a third Line $\alpha\varepsilon$, deſcribed in like manner, whoſe Fluent may be z, having its parts correſponding with the others, as $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, &c. there muſt be another Equation, either given or aſſumed, to aſcertain the rate of flowing, or the relation of z to the Correlate x. Or it will be the ſame thing, if in the two Equations the Fluents x, y, z, are any how promiſcuouſly involved. For theſe two Equations will limit and determine the Law of flowing in each Line. And we may likewiſe remove the Origin of the Fluent z to what point we pleaſe of the Line $\alpha\varepsilon$. And ſo if there were more Lines, or more Fluents.

59. To exemplify what has been ſaid by an eaſy inſtance. Thus inſtead of the Equation $\dot{y}=\dot{x}xy$, we may aſſume $\dot{y}=\dot{x}y+\dot{x}xy$, where the Origin of x is changed, or x is diminiſh'd by Unity; for $1+x$ is ſubſtituted inſtead of x. The lawfulneſs of which Re-

duction may be thus proved from the Principles of Analyticks. Make $x = 1 + z$, whence $\dot{x} = \dot{z}$, which ſhews, that x and z flow or increaſe alike. Subſtitute theſe inſtead of x and $\dot{x}$ in the Equation $\dot{y} = \dot{x}xy$, and it will become $\dot{y} = \dot{z}y + \dot{z}zy$. This differs in nothing elſe from the aſſumed Equation $\dot{y} = \dot{x}y + \dot{x}xy$, only that the Symbol x is changed into the Symbol z, which can make no real change in the argumentation. So that we may as well retain the ſame Symbols as were given at firſt, and, becauſe $z = x - 1$, we may as well ſuppoſe x to be diminiſh'd by Unity.

60, 61. The Equation expreſſing the Relation of the Fluents will at all times give any of their contemporaneous parts; for aſſuming different values of the Correlate Quantity, we ſhall thence have the correſponding different values of the Relate, and then by ſubtraction we ſhall obtain the contemporary differences of each. Thus if the given Equation were $y = x + \frac{1}{x}$, where x is ſuppos'd to be a quantity equably increaſing or decreaſing; make $x = 0, 1, 2, 3, 4, 5,$ &c. ſucceſſively, then $y =$ infinite, $2, 2\frac{1}{2}, 3\frac{1}{3}, 4\frac{1}{4}, 5\frac{1}{5},$ &c. reſpectively. And taking their differences, while x flows from 0 to 1, from 1 to 2, from 2 to 3, &c. y will flow from infinite to 2, from 2 to $2\frac{1}{2}$, from $2\frac{1}{2}$ to $3\frac{1}{3}$, &c. that is, their contemporaneous parts will be 1, 1, 1, 1, &c. and infinite, $\frac{1}{2}, \frac{5}{6}, \frac{11}{12}$, &c. reſpectively. Likewiſe, if we go backwards, or if we make x negative, we ſhall have $x = 0, -1, -2,$ &c. which will make $y =$ infinite, $-2, -2\frac{1}{2}$, &c. ſo that the contemporaneous differences will be as before.

Perhaps it may make a ſtronger impreſſion upon the Imagination, to repreſent this by a Figure. To the rectangular Aſymptotes GOH and KOL let ABC and DEF be oppoſite Hyperbola's; biſect the Angle GOK by the indefinite right Line vOR, perpendicular to which draw the Diameter BOE, meeting the Hyperbola's in B and E, from whence draw BQP and EST, as alſo CLR and DKU parallel to GOH. Now if OL is made to repreſent the indefinite and equable quantity x in the Equation $y = x + \frac{1}{x}$, then CR may repreſent y. For $\mathrm{CL} = \frac{1}{\mathrm{OL}} = \frac{1}{x}$, (ſuppoſing BQ $=$ OQ $= 1$,) and LR $=$ OL $= x$; therefore CR $=$ LR $+$ CL,



or $y = x + \frac{1}{x}$. Now the Origin of OL, or x, being in O; if $x = 0$, then CR, or y, will coincide with the Aſymptote OG, and therefore will be infinite. If $x = 1 =$ OQ, then $y =$ BP $= 2$. If $x = 2 =$ OL, then $y =$ CR $= 2\frac{1}{2}$. And ſo of the reſt. Alſo proceeding the contrary way, if $x = 0$, then y may be ſuppoſed to coincide with the Aſymptote OH, and therefore will be negative and infinite. If $x =$ OS $= -1$, then $y =$ ET $= -2$. If $x =$ OK $= -2$, then $y =$ Dv $= -2\frac{1}{2}$, &c. And thus we may purſue, at leaſt by Imagination, the correſpondent values of the flowing quantities x and y, as alſo their contemporary differences, through all their poſſible varieties; according to their relation to each other, as exhibited by the Equation $y = x + \frac{1}{x}$.

The Tranſition from hence to Fluxions is ſo very eaſy, that it may be worth while to proceed a little farther. As the Equation expreſſing the relation of the Fluents will give (as now obſerved) any of their contemporary parts or differences; ſo if theſe differences are taken very ſmall, they will be nearly as the Velocities of the moving Bodies, or points, by which they are deſcribed. For Motions continually accelerated or retarded, when perform'd in very ſmall ſpaces, become nearly equable Motions. But if thoſe differences are conceived to be diminiſhed *in infinitum*, ſo as from finite differences to become Moments, or vaniſhing Quantities, the Motions in them will be perfectly equable, and therefore the Velocities of their Deſcription, or the Fluxions of the Fluents, will be accurately as thoſe Moments. Suppoſe then x, y, z, &c. to repreſent Fluents in any Equation, or Equations, and their Fluxions, or Velocities of increaſe or decreaſe, to be repreſented by $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. and their reſpective contemporary Moments to be op, oq, or, &c. where p, q, r, &c. will be the Exponents of the Proportions of the Moments, and o denotes a vaniſhing quantity, as the nature of Moments requires. Then $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. will be as op, oq, or, &c. that is, as p, q, r, &c. So that $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. may be uſed inſtead of p, q, r, &c. in the deſignation of the Moments. That is, the ſynchronous Moments of x, y, z, &c. may be repreſented by $o\dot{x}$, $o\dot{y}$, $o\dot{z}$, &c. Therefore in any Equation the Fluent x may be ſuppoſed to be increaſed by its Moment $o\dot{x}$, and the Fluent y by its Moment $o\dot{y}$, &c. or $x + o\dot{x}$, $y + o\dot{y}$, &c. may be ſubſtituted in the Equation inſtead of x, y, &c. and yet the Equation will ſtill be true, becauſe the Moments are ſuppoſed to be ſynchronous. From which Operation an Equation will be form'd, which, by due Reduction, muſt neceſſarily exhibit the relation of the Fluxions.

Thus, for example, if the Equation $y = x + z$ be given, by Subſtitution we ſhall have $y + o\dot y = x + o\dot x + z + o\dot z$, which, becauſe $y = x + z$, will become $o\dot y = o\dot x + o\dot z$, or $\dot y = \dot x + \dot z$, which is the relation of the Fluxions. Here again, if we aſſume $z = \frac{1}{x}$, or $zx = 1$, by increaſing the Fluents by their contemporary Moments, we ſhall have $\overline{z + o\dot z} \times \overline{x + o\dot x} = 1$, or $zx + o\dot z x + o\dot x z + oo\dot z\dot x = 1$. Here becauſe $zx = 1$, 'tis $o\dot z x + o\dot x z + oo\dot z\dot x = 0$, or $\dot z x + \dot x z + o\dot z\dot x = 0$. But becauſe $o\dot z\dot x$ is a vaniſhing Term in reſpect of the others, 'tis $\dot z x + \dot x z = 0$, or $\dot z = -\frac{\dot x z}{x} = -\frac{\dot x}{x^2}$. Now as the Fluxion of z comes out negative, 'tis an indication that as x increaſes z will decreaſe, and the contrary. Therefore in the Equation $y = x + z$, if $z = \frac{1}{x}$, or if the relation of the Fluents be $y = x + \frac{1}{x}$, then the relation of the Fluxions will be $\dot y = \dot x - \frac{\dot x}{x^2}$.

And as before, from the Equation $y = x + \frac{1}{x}$ we derived the contemporaneous parts, or differences of the Fluents; ſo from the Fluxional Equation $\dot y = \dot x - \frac{\dot x}{x^2}$ now found, we may obſerve the rate of flowing, or the proportion of the Fluxions at different values of the Fluents.

For becauſe it is $\dot x : \dot y :: 1 : 1 - \frac{1}{x^2} :: x^2 : x^2 - 1$; when $x = 0$, or when the Fluent is but beginning to flow, (conſequently when y is infinite,) it will be $\dot x : \dot y :: 0 : -1$. That is, the Velocity wherewith x is deſcribed is infinitely little in compariſon of the velocity wherewith y is deſcribed; and moreover it is inſinuated, (becauſe of -1,) that while x increaſes by any finite quantity, tho' never ſo little, y will decreaſe by an infinite quantity at the ſame time. This will appear from the inſpection of the foregoing Figure. When $x = 1$, (and conſequently $y = 2$,) then $\dot x : \dot y :: 1 : 0$. That is, x will then flow infinitely faſter than y. The reaſon of which is, that y is then at its Limit, or the leaſt that it can poſſibly be, and therefore in that place it is ſtationary for a moment, or its Fluxion is nothing in compariſon of that of x. So in the foregoing Figure, BP is the leaſt of all ſuch Lines as are repreſented by CR. When $x = 2$, (and therefore $y = 2\frac{1}{2}$,) it will be $\dot x : \dot y :: 4 : 3$. Or the Velocity of x is there greater than that of y, in the ratio of 4 to 3. When $x = 3$, then $\dot{x} : \dot{y} :: 9 : 8$. And ſo on. So that the Velocities or Fluxions conſtantly tend towards equality, which they do not attain till $\frac{1}{x}$ (or CL) finally vaniſhing, x and y become equal. And the like may be obſerved of the negative values of x and y.

SECT. V. *The Reſolution of Equations, whether Algebraical or Fluxional, by the aſſiſtance of ſuperior orders of Fluxions.*

ALL the foregoing Extractions (according to a hint of our Author's,) may be perform'd ſomething more expeditiouſly, and without the help of ſubſidiary Operations, if we have recourſe to ſuperior orders of Fluxions. To ſhew this firſt by an eaſy Inſtance.

Let it be required to extract the Cube-root of the Binomial $a^3 + x^3$, or to find the Root y of this Equation $y^3 = a^3 + x^3$; or rather, for ſimplicity-ſake, let it be $y^3 = a^3 + z$. Then $y = a$, &c. or the initial Term of y will be a. Taking the Fluxions of this Equation, we ſhall have $3\dot{y}y^2 = \dot{z} = 1$, or $\dot{y} = \frac{1}{3}y^{-2}$. But as it is $y = a$, &c. by ſubſtitution it will be $\dot{y} = \frac{1}{3}a^{-2}$, &c. and taking the Fluents, 'tis $y = * + \frac{1}{3}a^{-2}z$, &c. Here a vacancy is left for the firſt Term of y, which we already know to be a. For another Operation take the Fluxions of the Equation $\dot{y} = \frac{1}{3}y^{-2}$; whence $\ddot{y} = -\frac{2}{3}\dot{y}y^{-3} = -\frac{2}{9}y^{-5}$. Then becauſe $y = a$, &c. 'tis $\ddot{y} = -\frac{2}{9}a^{-5}$, &c. and taking the Fluents, 'tis $\dot{y} = * - \frac{2}{9}a^{-5}z$, &c. and taking the Fluents again, 'tis $y = * * - \frac{1}{9}a^{-5}z^2$, &c. Here two vacancies are to be left for the two firſt Terms of y, which are already known. For the next Operation take the Fluxions of the Equation $\ddot{y} = -\frac{2}{9}y^{-5}$, that is, $\dddot{y} = +\frac{10}{9}\dot{y}y^{-6} = +\frac{10}{27}y^{-8}$. Or becauſe $y = a$, &c. 'tis $\dddot{y} = \frac{10}{27}a^{-8}$, &c. Then taking the Fluents, 'tis $\ddot{y} = * \frac{10}{27}a^{-8}z$, &c. $\dot{y} = * * \frac{5}{27}a^{-8}z^2$, &c. and $y = * * * \frac{5}{81}a^{-8}z^3$, &c. Again, for another Operation take the Fluxions of the Equation $\dddot{y} = \frac{10}{27}y^{-8}$; whence $\ddddot{y} = -\frac{80}{27}\dot{y}y^{-9} = -\frac{80}{81}y^{-11}$. Or becauſe $y = a$, &c. 'tis $\ddddot{y} = -\frac{80}{81}a^{-11}$, &c. Then taking the Fluents, $\dddot{y} = * - \frac{80}{81}a^{-11}z$, &c. $\ddot{y} = * * - \frac{40}{81}a_{-11}z^2$,

$\frac{40}{81}a^{-11}z^2$, &c. $\dot{y} = * * * - \frac{40}{243}a^{-11}z^3$, &c. and $y = * * * * - \frac{10}{243}a^{-11}z^4$, &c. And ſo we may go on as far as we pleaſe. We have therefore found at laſt, that $y = a + \frac{z}{3a^2} - \frac{z^2}{9a^5} + \frac{5z^3}{81a^8} - \frac{10z^4}{243a^{11}}$, &c. or for z writing x^3, 'tis $\sqrt[3]{a^3 + x^3} = a + \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}}$, &c.

Or univerſally, if we would reſolve $\overline{a+x}\,|^{m}$ into an equivalent infinite Series, make $y = \overline{a+x}\,|^{m}$, and we ſhall have a^m for the firſt Term of the Series y, or it will be $y = a^m$, &c. Then becauſe $y^{\frac{1}{m}} = a + x$, taking the Fluxions we ſhall have $\frac{1}{m}\dot{y}y^{\frac{1}{m}-1} = \dot{x} = 1$, or $\dot{y} = my^{1-\frac{1}{m}}$. But becauſe it is $y = a^m$, &c. it will be $\dot{y} = ma^{m-1}$, &c. and now taking the Fluents, 'tis $y = * ma^{m-1}x$, &c. Again, becauſe it is $\dot{y} = my^{1-\frac{1}{m}}$, taking the Fluxions it will be $\ddot{y} = \overline{m-1}\dot{y}y^{-\frac{1}{m}} = m \times \overline{m-1}y^{1-\frac{2}{m}}$; and becauſe $y = a^m$, &c. 'tis $\ddot{y} = m \times \overline{m-1}a^{m-2}$, &c. And taking the Fluents, 'tis $\dot{y} = * m \times \overline{m-1}a^{m-2}x$, &c. and therefore $y = * * m \times \frac{m-1}{2}a^{m-2}x^2$, &c. Again, becauſe it is $\ddot{y} = m \times \overline{m-1}y^{1-\frac{2}{m}}$, taking the Fluxions it will be $\dddot{y} = \overline{m-1} \times \overline{m-2}\dot{y}y^{-\frac{2}{m}} = m \times \overline{m-1} \times \overline{m-2}y^{1-\frac{3}{m}}$; and becauſe $y = a^m$, &c. 'tis $\dddot{y} = m \times \overline{m-1} \times \overline{m-2}a^{m-2}$, &c. And taking the Fluents, 'tis $\ddot{y} = * m \times \overline{m-1} \times \overline{m-2}a^{m-3}x$, &c. $\dot{y} = * * m \times \frac{m-1}{2} \times \overline{m-2}a^{m-3}x^3$, &c. and $y = * * * m \times \frac{m-1}{2} \times \frac{m-2}{3}a^{m-3}x^3$, &c. And ſo we might proceed as far as we pleaſe, if the Law of Continuation had not already been ſufficiently manifeſt. So that we ſhall have here $\overline{a+x}\,|^{m} = a^m + ma^{m-1}x + m \times \frac{m-1}{2}a^{m-2}x^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3}a^{m-3}x^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}a^{m-4}x^4$, &c.

This is a famous Theorem of our Author's, tho' diſcover'd by him after a very different manner of Inveſtigation, or rather by Induction. It is commonly known by the name of his Binomial Theorem, becauſe by its aſſiſtance any Binomial, as $a+x$, may be raiſed to any Power at pleaſure, or any Root of it may be extracted. And it is obvious, that when m is interpreted by any integer affirmative Number, the Series will break off, and become finite, at a number of Terms denominated by m. But in all other caſes it will be an infinite Series, which will converge when x is leſs than a.

Indeed it can hardly be ſaid, that this, or any other that is derived from the Method of Fluxions, is a ſtrict Inveſtigation of this Theorem. Becauſe that Method itſelf is originally derived from the Method of raiſing Powers, at leaſt integral Powers, and previouſly ſuppoſes the knowledge of the *Unciæ*, or the numeral Coefficients. However it may anſwer the intention, of being a proper Example of this Method of Extraction, which is all that is neceſſary here.

There is another Theorem for this purpoſe, which I found many years ago, and then communicated it to my ingenious Friend Mr. *A. de Moivre*, who liked it ſo well as to inſert it in a Mathematical Treatiſe he was then publiſhing. I ſhall here give the Reader its Inveſtigation, in the ſame manner it was found.

Let us ſuppoſe $\overline{a+x}|^m = a^m + p$, and that $a+x = z$, and therefore $\dot{z} = \dot{x} = 1$. Now becauſe $z^m = a^m + p$, it will be $\dot{p} = mz^{m-1} = \frac{mz^m}{z}$; where for z^m writing its value $a^m + p$, we ſhall have $\dot{p} = \frac{ma^m}{z} + \frac{mp}{z}$. Now if we make $p = \frac{ma^m x}{z} + q$, it will be $\dot{p} = \frac{ma^m}{z} - \frac{ma^m x}{z^2} + \dot{q}$. And comparing theſe two values of $\dot{p}$, we ſhall have $\dot{q} = \frac{ma^m x}{z^2} + \frac{mp}{z}$; where if for p we write its value as above, it will be $\dot{q} = \frac{ma^m x}{z^2} + \frac{m^2 a^m x}{z^2} + \frac{mq}{z}$, or $\dot{q} = m \times \overline{m+1} \times \frac{a^m x}{z^2} + \frac{mq}{z}$; make $q = m \times \frac{m+1}{2} \times \frac{a^m x^2}{z^2} + r$; therefore $\dot{q} = m \times \overline{m+1} \times \frac{a^m x}{z^2} - m \times \overline{m+1} \times \frac{a^m x^2}{z^3} + \dot{r}$. From which two values of $\dot{q}$ we ſhall have $\dot{r} = m \times \overline{m+1} \times \frac{a^m x^2}{z^3} + \frac{mq}{z}$. And for q ſubſtituting its value, it will be $\dot{r} = m \times \overline{m+1} \times \frac{a^m x^2}{z^3} + m^2 \times \frac{m+1}{2} \times \frac{a^m x^2}{z^3} + \frac{mr}{z}$. Or $\dot{r} = m \times \frac{m+1}{2} \times \frac{m+2}{1} \times \frac{a^m x^2}{z^3} + \frac{mr}{z}$. Make $r = m \times \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{a^m x^3}{z^3} + s$; then, &c. So that we ſhall have $\overline{a+x}|^{m} = a^{m} + m \times \frac{a^{m}x}{a+x} + m \times \frac{m+1}{2}\,\frac{a^{m}x^{2}}{\overline{a+x}|^{2}} + m \times \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{a^{m}x^{3}}{\overline{a+x}|^{3}}$, &c.

Now this Series will ſtop of its own accord, at a finite number of Terms, when m is any integer and negative Number; that is, when the Reciprocal of any Power of a Binomial is to be found. But in all other caſes we ſhall have an infinite converging Series for the Power or Root required, which will always converge when a and x have the ſame Sign; becauſe the Root of the Scale, or the converging quantity, is $\frac{x}{a+x}$; which is always leſs than Unity.

By comparing theſe two Series together, or by collecting from each the common quantity $\frac{\overline{a+x}|^{m} - a^{m}}{ma^{m}x}$, we ſhall have the two equivalent Series $\frac{1}{a} + \frac{m-1}{2} \times \frac{x}{a^{2}} + \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{x^{2}}{a^{3}}$, &c. $= \frac{1}{a+x} + \frac{m+1}{2} \times \frac{x}{\overline{a+x}|^{2}} + \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{x^{2}}{\overline{a+x}|^{3}}$, &c. from whence we might derive an infinite number of Numeral Converging Series, not inelegant, which would be proper to explain and illuſtrate the nature of Convergency in general, as has been attempted in the former part of this work. For if we aſſume ſuch a value of m as will make either of the Series become finite, the other Series will exhibit the quantity that ariſes by an Approximation *ad infinitum*. And then a and x may be afterwards determined at pleaſure.

As another Example of this Method, we ſhall ſhew (according to promiſe) how to derive Mr. *de Moivre*'s elegant Theorem; for raiſing an Infinitinomial to any indeterminate Power, or for extracting any Root of the ſame. The way how it was derived from the abſtract conſideration of the nature and geneſis of Powers, (which indeed is the only legitimate method of Inveſtigation in the preſent caſe,) and the Law of Continuation, have been long ago communicated and demonſtrated by the Author, in the Philoſophical Tranſactions, N° 230. Yet for the dignity of the Problem, and the better to illuſtrate the preſent Method of Extraction of Roots, I ſhall deduce it here as follows.

Let us aſſume the Equation $\overline{a + bz + cz^{2} + dz^{3} + ez^{4}, \&c.}|^{m} = y$, where the value of y is to be found by an infinite Series, of which the firſt Term is already known to be a^{m}, or it is $y = a^{m}$, &c. Make $v = a + bz + cz^{2} + dz^{3} + ez^{4}$, &c. and putting $\dot{z} = 1$, and taking the Fluxions, we ſhall have $\dot{v} = b + 2cz +$

$3dz^2 + 4ez^3$, &c. Then becauſe $y = v^m$, it is $\dot{y} = m\dot{v}v^{m-1}$, where if we make $v = a$, &c. and $\dot{v} = b$, &c. we ſhall have $\dot{y} = ma^{m-1}b$, &c. and taking the Fluents, it will be $y = * \; ma^{m-1}bz$, &c.

For another Operation, becauſe $\dot{y} = m\dot{v}v^{m-1}$, it is $\ddot{y} = m\ddot{v}v^{m-1} + m \times \overline{m-1}\dot{v}^2v^{m-2}$. And becauſe $\ddot{v} = 2c + 6dz + 12ez^2$, &c. for v, $\dot{v}$, and $\ddot{v}$ ſubſtituting their values a, &c. b, &c. and $2c$, &c. reſpectively, we ſhall have $\ddot{y} = 2mca^{m-1} + m \times \overline{m-1}b^2a^{m-2}$, &c. and taking the Fluents $\dot{y} = * \; 2mca^{m-1}z + m \times \overline{m-1}b^2a^{m-2}z$, &c. and taking the Fluents again, $y = * * mca^{m-1}z^2 + m \times \frac{m-1}{2}b^2a^{m-2}z^2$, &c.

For another Operation, becauſe $\ddot{y} = m\ddot{v}v^{m-1} + m \times \overline{m-1}\dot{v}^2v^{m-2}$, 'tis $\dddot{y} = m\dddot{v}v^{m-1} + 3m \times \overline{m-1}v^{m-2}\ddot{v}\dot{v} + m \times \overline{m-1} \times \overline{m-2}v^{m-3}\dot{v}^3$. And becauſe $\dddot{v} = 6d + 24ez$, &c. for v, $\dot{v}$, $\ddot{v}$, $\dddot{v}$, ſubſtituting a, &c. b, &c. $2c$, &c. $6d$, &c. we ſhall have $\dddot{y} = 6mda^{m-1} + 6m \times \overline{m-1}bca^{m-2} + m \times \overline{m-1} \times \overline{m-2}b^3a^{m-3}$, &c. And taking the Fluents it will be $\ddot{y} = * \; 6mda^{m-1}z + 6m \times \overline{m-1}bca^{m-2}z + m \times \overline{m-1} \times \overline{m-2}b^3a^{m-3}z$, &c. $\dot{y} = * * 3mda^{m-1}z^2 + 3m \times \overline{m-1}bca^{m-2}z^2 + m \times \frac{m-1}{2} \times \overline{m-2}b^3a^{m-3}z^2$, &c. and $y = * * * mda^{m-1}z^3 + m \times \frac{m-1}{1}bca^{m-2}z^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3}b^3a^{m-3}z^3$, &c. And ſo on *in infinitum*. We ſhall therefore have $\overline{a + bz + cz^2 + dz^3 + ez^4, \&c.}|^m =$

$$\begin{array}{lll} (y=)a^m + ma^{m-1}bz & + m \times \frac{m-1}{2}a^{m-2}b^2 \times z^2 & + m \times \frac{m-1}{2} \times \frac{m-2}{3}a^{m-3}b^3 \times z^3, \&c. \\ & + ma^{m-1}c & + m \times \frac{m-1}{1}a^{m-2}bc \\ & & + ma^{m-1}d \end{array}$$

And if the whole be multiply'd by z^n, and continued to a due length, it will have the form of Mr. *de Moivre*'s Theorem.

The Roots of all Algebraical or Fluential Equations may be extracted by this Method. For an Example let us take the Cubick Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, ſo often before reſolved, in which $y = a$, &c. Then taking the Fluxions, and making $\dot{x} = 1$, we ſhall have $3\dot{y}y^2 + ay + ax\dot{y} + a^2\dot{y} - 3x^2 = 0$. Here if for y we ſubſtitute a, &c. we ſhall have $4a^2\dot{y} + a^2 + ax\dot{y} - 3x^2$, &c. $= 0$, or $\dot{y} = \frac{-a^2 + 3x^2, \&c.}{4a^2 + ax, \&c.} = \frac{-a^2}{4a^2}$, &c. $= -\frac{1}{4}$, &c. And taking the Fluents, $y = * - \frac{1}{4}x$, &c. Then taking the Fluxions again of the laſt Equation, we ſhall have $3\ddot{y}y^2 + 6\dot{y}^2y + 2a\dot{y} + ax\ddot{y} + a^2\ddot{y} - 6x = 0$. Where if we make $y = a$, &c. and $\dot{y} = -\frac{1}{4}$, &c. we ſhall have $\ddot{y} = \frac{-\frac{3}{8}a + \frac{1}{2}a, \text{ \&c.}}{4a^2, \text{ \&c.}} = \frac{1}{32a}$, &c. and therefore $\dot{y} = * + \frac{x}{32a}$, &c. and $y = * * + \frac{x^2}{64a}$, &c. Again, $3\dddot{y}y^2 + 18\ddot{y}\dot{y}y + 6\dot{y}^3 + 3a\ddot{y} + ax\dddot{y} + a^2\dddot{y} - 6 = 0$. Make $y = a$, &c. $\dot{y} = -\frac{1}{4}$, &c. and $\ddot{y} = \frac{1}{32a}$, &c. then $\dddot{y} = \frac{\frac{9}{64} + \frac{6}{64} - \frac{3}{32} + 6}{4a^2}$, &c. $= \frac{393}{256a^2}$, &c. and therefore $\ddot{y} = * \frac{393x}{256a^2}$, &c. $\dot{y} = * * \frac{393x^2}{512a^2}$, &c. and $y = * * * \frac{131x^3}{512a^2}$, &c. Again, $3\ddddot{y}y^2 + 24\dddot{y}\dot{y}y + 18\ddot{y}^2y + 36\ddot{y}\dot{y}^2 + 4a\dddot{y} + ax\ddddot{y} + a^2\ddddot{y} = 0$. Make $y = a$, &c. $\dot{y} = -\frac{1}{4}$, &c. $\ddot{y} = \frac{1}{32a}$, &c. and $\dddot{y} = \frac{393}{256a^2}$, &c. then $\ddddot{y} = -\frac{24\dddot{y}\dot{y}y + 18\ddot{y}^2y + 36\ddot{y}\dot{y}^2 + 4a\dddot{y}}{3y^2 + a^2, \text{ \&c.}} = \frac{1527}{2048a^3}$, &c. and $\dddot{y} = * \frac{1527x}{2048a^3}$, &c. $\ddot{y} = * * \frac{1527x^2}{4096a^3}$, &c. $\dot{y} = * * * \frac{509x^3}{4096a^3}$, &c. and $y = * * * * \frac{509x^4}{16384a^3}$, &c. And ſo on as far as we pleaſe. Therefore the Root is $y = a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$, &c.

The Series for the Root, when found by this Method, muſt always have its Powers aſcending; but if we deſire likewiſe to find a Series with deſcending Powers, it may be done by this eaſy artifice. As in the preſent Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, we may conceive x to be a conſtant quantity, and a to be a flowing quantity; or rather, to prevent a confuſion of Ideas, we may change a into x, and x into a, and then the Equation will be $y^3 + axy + x^2y - a^3 - 2x^3 = 0$. In this we ſhall have $y = a$, &c. and taking the Fluxions, 'tis $3\dot{y}y^2 + ay + ax\dot{y} + 2xy + x^2\dot{y} - 6x^2 = 0$, or $\dot{y} = \frac{-ay - 2xy + 6x^2}{3y^2 + ax + x^2}$. But becauſe $y = a$, &c. 'tis $\dot{y} = \frac{-aa}{3aa}$, &c. $= -\frac{1}{3}$, &c. and therefore $y = * - \frac{1}{3}x$, &c. Again taking the Fluxions 'tis $3\ddot{y}y^2 + 6\dot{y}^2y + 2a\dot{y} + ax\ddot{y} + 2y + 4x\dot{y} + x^2\ddot{y} - 12x = 0$, or $\ddot{y} = \frac{-6\dot{y}^2y - 2a\dot{y} - 2y - 4x\dot{y} + 12x}{3y^2 + ax + x^2} = \frac{-6\dot{y}^2y - 2a\dot{y} - 2y}{3y^2}$, &c. Or making $y = a$, &c. and $\dot{y} = -\frac{1}{3}$, &c. 'tis $\ddot{y} = \frac{-\frac{2}{3}a + \frac{2}{3}a - 2a}{3a^2}$, &c. $= -\frac{2}{3a}$, &c. and $\dot{y} = * - \frac{2x}{3a}$, &c. and $y = * * - \frac{x^2}{3a}$, &c. Again it is $3\dddot{y}y^2 + 18\ddot{y}\dot{y}y + 6\dot{y}^3 + 3a\ddot{y} + ax\dddot{y} + 6\dot{y} + 6x\ddot{y} + x^2\dddot{y}$

$-12 = 0$, or $\dddot{y} = \frac{-18\ddot{y}\dot{y}y - 6\dot{y}^3 - 3a\ddot{y} - 6\dot{y} + 12}{3y^2}$, &c. $=$ (by making $y = a$, &c. $\dot{y} = -\frac{1}{3}$, &c. and $\ddot{y} = -\frac{2}{3a}$, &c.) $\frac{-4 + \frac{2}{9} + 2 + 2 + 12}{3a^2}$, &c. $= \frac{110}{27a^2}$, &c. Then taking the Fluents, $\ddot{y} = * \frac{110x}{27a^2}$, &c. $\dot{y} = * * \frac{55x^2}{27a^2}$, &c. and $y = * * * \frac{55x^3}{81a^2}$, &c. And ſo on. Therefore we ſhall have $y = a - \frac{1}{3}x - \frac{x^2}{3a} + \frac{55x^3}{81a^2}$, &c. Or now we may again change x into a, and a into x; then it will be $y = x - \frac{1}{3}a - \frac{a^2}{3x} + \frac{55a^3}{81x^2}$, &c. for the Root of the given Equation, as was found before, pag. 216, &c.

Alſo in the Solution of Fluxional Equations, we may proceed in the ſame manner. As if the given Equation were $a^2\dot{y} - a^2\dot{x} + x^2\dot{y} = 0$, (in which, if the Radius of a Circle be repreſented by a, and if y be any Arch of the ſame, the correſponding Tangent will be repreſented by x;) let it be required to extract the Root y out of this Equation, or to expreſs it by a Series compoſed of the Powers of a and x. Make $\dot{x} = 1$, then the Equation will be $a^2\dot{y} - a^2 + x^2\dot{y} = 0$. Here becauſe $\dot{y} = \frac{a^2}{a^2 + x^2} = 1$, &c. taking the Fluents it will be $y = * x$, &c. Then taking the Fluxions of this Equation, we ſhall have $a^2\ddot{y} + 2x\dot{y} + x^2\ddot{y} = 0$, or $\ddot{y} = -\frac{2x\dot{y}}{a^2 + x^2}$. But becauſe we are to have a conſtant quantity for the firſt Term of $\ddot{y}$, we may ſuppoſe $\ddot{y} = \frac{0 - 2x\dot{y}}{a^2 + x^2} = 0$, &c. Then taking the Fluents 'tis $\dot{y} = * 0$, &c. and $y = * * 0$, &c. Then taking the Fluxions again, 'tis $a^2\dddot{y} + 2\dot{y} + 4x\ddot{y} + x^2\dddot{y} = 0$, or $\dddot{y} = \frac{-2\dot{y} - 4x\ddot{y}}{a^2 + x^2}$. Here if for $\dot{y}$ and $\ddot{y}$ we write their values 1, &c. and 0, &c. we ſhall have $\dddot{y} = -\frac{2}{a^2}$, &c. whence $\ddot{y} = * -\frac{2x}{a^2}$, &c. $\dot{y} = * * -\frac{x^2}{a^2}$, &c. and $y = * * * -\frac{x^3}{3a^2}$, &c. Taking the Fluxions again, 'tis $a^2\ddddot{y} + 6\ddot{y} + 6x\dddot{y} + x^2\ddddot{y} = 0$, or $\ddddot{y} = \frac{-6\ddot{y} - 6x\dddot{y}}{a^2 + x^2} = 0$, &c. Therefore $\dddot{y} = * 0$, &c. $\ddot{y} = * * 0$, &c. $\dot{y} = * * * 0$, &c. $y = * * * * 0$, &c. Again, $a^2\overset{5}{y} + 12\dddot{y} + 8x\ddddot{y} + x^2\overset{5}{y} = 0$, or $\overset{5}{y} = \frac{-12\dddot{y} - 8x\ddddot{y}}{a^2 + x^2}$

$=$

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$= + \frac{24}{a^4}$, &c. Then $\ddddot{y} = * + \frac{24x}{a^4}$, &c. $\dddot{y} = * * + \frac{12x^2}{a^4}$, &c. $\ddot{y} = * * * + \frac{4x^3}{a^4}$, &c. $\dot{y} = * * * * + \frac{x^4}{a^4}$, &c. and $y = * * * * * + \frac{x^5}{5a^4}$, &c. Again, $a^2\overset{6}{\dot{y}} + 20\overset{4}{\dot{y}} + 10x\overset{5}{\dot{y}} + x^2\overset{6}{\dot{y}} = 0$, whence $y = * * * * * * * 0$, &c. Again, $a^2\overset{7}{\dot{y}} + 30\overset{5}{\dot{y}} + 12x\overset{6}{\dot{y}} + x^2\overset{7}{\dot{y}} = 0$, or $\overset{7}{\dot{y}} = \frac{-30\overset{5}{\dot{y}} - 12x\overset{6}{\dot{y}}}{a^2 + x^2} = -30 \times 24a^{-6}$, &c. Then $\overset{6}{\dot{y}} = * - \frac{24 \times 30x}{a^6}$, &c. $\overset{5}{\dot{y}} = * * - \frac{12 \times 30x^2}{a^6}$, &c. $\overset{4}{\dot{y}} = * * * - \frac{4 \times 30x^3}{a^6}$, &c. $\dddot{y} = * * * * - \frac{30x^4}{a^6}$, &c. $\ddot{y} = * * * * * - \frac{6x^5}{a^6}$, &c. $\dot{y} = * * * * * * - \frac{x^6}{a^6}$, &c. and $y = * * * * * * * - \frac{x^7}{7a^6}$, &c. And ſo on. So that we have here $y = * \; x + 0x^2 - \frac{x^3}{3a^2} + 0x^4 + \frac{x^5}{5a^4}$, &c. that is, $y = x - \frac{x^3}{3a^2} + \frac{x^5}{5a^4} - \frac{x^7}{7a^6}$, &c.

This Example is only to ſhew the univerſality of this Method, and how we are to proceed in other like caſes; for as to the Equation itſelf, it might have been reſolved much more ſimply and expeditiouſly, in the following manner. Becauſe $\dot{y} = \frac{a^2}{a^2 + x^2}$, by Diviſion it will be $\dot{y} = 1 - \frac{x^2}{a^2} + \frac{x^4}{a^4} - \frac{x^6}{a^6} + \frac{x^8}{a^8}$, &c. And taking the Fluents, $y = x - \frac{x^3}{3a^2} + \frac{x^5}{5a^4} - \frac{x^7}{7a^6} + \frac{x^9}{9a^8}$, &c.

In the ſame Equation $a^2\dot{y} - a^2\dot{x} + x^2\dot{y} = 0$, if it were requir'd to expreſs x by y, (the Tangent by the Arch,) or if x were made the Relate, and y the Correlate, we might proceed thus. Make $\dot{y} = 1$, then $a^2 - a^2\dot{x} + x^2 = 0$, or $\dot{x} = 1 + \frac{x^2}{a^2} = 1$, &c. Then $x = * \, y$, &c. And taking the Fluxions, 'tis $\ddot{x} = \frac{2x\dot{x}}{a^2} = \frac{2x}{a^2}$, &c. $= 0 + \frac{2x}{a^2}$, &c. whence $\dot{x} = * \, 0$, &c. and $x = * * \, 0$, &c. So that the Terms of this Series will be alternately deficient, and therefore we need not compute them. Taking the Fluxions again, 'tis $\dddot{x} = \frac{2\dot{x}^2}{a^2} + \frac{2x\ddot{x}}{a^2} = \frac{2}{a^2}$, &c. Therefore $\ddot{x} = * \, \frac{2y}{a^2}$, &c. $\dot{x} = * * \, \frac{y^2}{a^2}$, &c. and $x = * * * \, \frac{y^3}{3a^2}$, &c. Again, $\ddddot{x} = \frac{6\dot{x}\ddot{x}}{a^2} + \frac{2x\dddot{x}}{a^2}$, and again, $\overset{5}{x} = \frac{6\ddot{x}^2}{a^2} + \frac{8\dot{x}\dddot{x}}{a^2} + \frac{2x\ddddot{x}}{a^2}$. Subſtituting 1, &c. and $\frac{2}{a^2}$, &c. for $\dot{x}$ and $\dddot{x}$, and alſo 0, &c. for $\ddot{x}$ and $\ddddot{x}$, it will be $\overset{5}{x} = \frac{16}{a^4}$, &c. whence $\ddddot{x} = * \frac{16y}{a^4}$, &c. $\dddot{x} = * * \frac{8y^2}{a^4}$, &c. $\ddot{x} = * * * \frac{8y^3}{3a^2}$, &c. $\dot{x} = * * * * \frac{2y^4}{3a^4}$, &c. and $x = * * * * * \frac{2y^5}{15a^4}$, &c. Again, $\overset{6}{x} = \frac{20\ddot{x}\dddot{x} + 10\dot{x}\ddddot{x} + 2x\overset{5}{x}}{a^2}$, and again, $\overset{7}{x} = \frac{20\dddot{x}^2 + 30\ddot{x}\ddddot{x} + 12\dot{x}\overset{5}{x} + 2x\overset{6}{x}}{a^2}$. Here for $\dot{x}$, $\dddot{x}$, and $\overset{5}{x}$ writing 1, &c. $\frac{2}{a^2}$, &c. and $\frac{16}{a^4}$, &c. reſpectively, 'tis $\overset{7}{x} = \frac{80 + 12 \times 16}{a^6}$, &c. $= \frac{272}{a^6}$, &c. Then $\overset{6}{x} = * \frac{272y}{a^6}$, &c. $\overset{5}{x} = * * \frac{136y^2}{a^6}$, &c. $\overset{4}{x} = * * * \frac{136y^3}{3a^6}$, &c. $\dddot{x} = * * * * \frac{34y^4}{3a^6}$, &c. $\ddot{x} = * * * * * \frac{34y^5}{15a^6}$, &c. $\dot{x} = * * * * * * \frac{17y^6}{45a^6}$, &c. and $x = * * * * * * * \frac{17y^7}{315a^6}$, &c. That is, $x = y + \frac{y^3}{3a^2} + \frac{2y^5}{15a^4} + \frac{17y^2}{315a^6}$, &c.

For another Example, let us take the Equation $a^2\dot{y}^2 - x^2\dot{y}^2 - a^2\dot{x}^2 = 0$, (in which, if the Radius of a Circle be denoted by a, and if y be any Arch of the ſame, then the correſponding right Sine will be denoted by x;) from which we are to extract the Root y. Make $\dot{x} = 1$, then it will be $a^2\dot{y}^2 - x^2\dot{y}^2 = a^2$, or $\dot{y}^2 = \frac{a^2}{a^2 - x^2} = 1$, &c. or $\dot{y} = 1$, &c. and therefore $y = * x$, &c. Taking the Fluxions we ſhall have $2a^2\dot{y}\ddot{y} - 2x\dot{y}^2 - 2x^2\dot{y}\ddot{y} = 0$, or $a^2\ddot{y} - x\dot{y} - x^2\ddot{y} = 0$, or $\ddot{y} = \frac{x\dot{y}}{a^2 - x^2} = 0$, &c. And taking the Fluxions again, 'tis $a^2\dddot{y} - \dot{y} - 3x\ddot{y} - x^2\dddot{y} = 0$, or $\dddot{y} = \frac{\dot{y} + 3x\ddot{y}}{a^2 - x^2} = \frac{1}{a^2}$, &c. Therefore $\ddot{y} = * \frac{x}{a^2}$, &c. $\dot{y} = * * \frac{x^2}{2a^2}$, &c. and $y = * * * \frac{x^3}{6a^2}$, &c. Then $a^2\ddddot{y} - 4\ddot{y} - 5x\dddot{y} - x^2\ddddot{y} = 0$, and again $a^2\overset{5}{y} - 9\dddot{y} - 7x\ddddot{y} - x^2\overset{5}{y} = 0$, or $\overset{5}{y} = \frac{9\dddot{y} + 7x\ddddot{y}}{a^2 - x^2} = \frac{9\dddot{y}}{a^2}$, &c. $= \frac{9}{a^4}$, &c. Therefore $\ddddot{y} = * \frac{9x}{a^4}$, &c. $\dddot{y} = * * \frac{9x^2}{2a^4}$, &c. $\ddot{y} = * * * \frac{3x^3}{2a^4}$, &c. $\dot{y} =$

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$* * * * \frac{3x^4}{8a^4}$, &c. and $y = * * * * * \frac{3x^5}{40a^4}$, &c. Taking the Fluxions again, 'tis $a^2\overset{6}{\dot{y}} - 16\ddddot{y} - 9x\overset{5}{\dot{y}} - x^2\overset{6}{\dot{y}} = 0$, and again, $a^2\overset{7}{\dot{y}} - 25\overset{5}{\dot{y}} - 11x\overset{6}{\dot{y}} - x^2\overset{7}{\dot{y}} = 0$, or $\overset{7}{\dot{y}} = \frac{25\overset{5}{\dot{y}} + 11x\overset{6}{\dot{y}}}{a^2 - x^2} = \frac{25}{a^2}\dot{y}^5$, &c. $= \frac{25\times 9}{a^6}$, &c. Therefore $\overset{6}{\dot{y}} = * \frac{25\times 9}{a^6}x$, &c. $\overset{5}{\dot{y}} = * * \frac{25\times 9}{2a^6}x^2$, &c. $\overset{4}{\dot{y}} = * * * \frac{25\times 3}{2a^6}x^3$, &c. $\dddot{y} = * * * * \frac{25\times 3}{8a^6}x^4$, &c. $\ddot{y} = * * * * * \frac{5\times 3}{8a^6}x^5$, &c. $\dot{y} = * * * * * * \frac{5x^6}{16a^6}$, &c. and $y = * * * * * * * \frac{5x^7}{112a^6}$, &c. Or $y = x + \frac{x^3}{6a^2} + \frac{3x^5}{40a^4} + \frac{5x^7}{112a^6}$, &c.

If we were required to extract the Root x out of the ſame Equation, $a^2\dot{y}^2 - x^2\dot{y}^2 - a^2\dot{x}^2 = 0$, (or to expreſs the Sine by the Arch,) put $\dot{y} = 1$, then $a^2 - x^2 - a^2\dot{x}^2 = 0$, or $\dot{x}^2 = 1 - \frac{x^2}{a^2}$, and therefore $\dot{x} = 1$, &c. and $x = * y$, &c. Taking the Fluxions 'tis $-2\dot{x}x - 2a^2\dot{x}\ddot{x} = 0$, or $\ddot{x} = -\frac{x}{a^2} = 0$, &c. Therefore $\dot{x} = * 0$, &c. $x = * * 0$, &c. Taking the Fluxions again, 'tis $\dddot{x} = -\frac{\dot{x}}{a^2} = -\frac{1}{a^2}$, &c. Thence $\ddot{x} = * -\frac{y}{a^2}$, &c. $\dot{x} = * * -\frac{y^2}{2a^2}$, &c. and $x = * * * -\frac{y^3}{6a^2}$, &c. Again, $\ddddot{x} = -\frac{\ddot{x}}{a^2}$, and $\overset{5}{\dot{x}} = -\frac{\dddot{x}}{a^2} = +\frac{1}{a^4}$, &c. Therefore $\ddddot{x} = * \frac{y}{a^4}$, &c. $\dddot{x} = * * \frac{y^2}{2a^4}$, &c. $\ddot{x} = * * * \frac{y^3}{6a^4}$, &c. $\dot{x} = * * * * \frac{y^4}{24a^4}$, &c. and $x = * * * * * \frac{y^5}{120a^4}$, &c. Again, $\overset{6}{\dot{x}} = -\frac{\ddddot{x}}{a^2}$, and $\overset{7}{\dot{x}} = -\frac{\overset{5}{\dot{x}}}{a^2} = -\frac{1}{a^6}$, &c. Therefore $\overset{6}{\dot{x}} = * -\frac{y}{a^6}$, $\overset{5}{\dot{x}} = * * -\frac{y^2}{2a^6}$, &c. $\ddddot{x} = * * * -\frac{y^3}{6a^6}$, &c. $\dddot{x} = * * * * -\frac{y^4}{24a^6}$, &c. $\ddot{x} = * * * * * -\frac{y^5}{120a^6}$, &c. $\dot{x} = * * * * * * -\frac{y^6}{720a^6}$, &c. and $x = * * * * * * * -\frac{y^7}{5040a^6}$, &c. And therefore $x = y - \frac{y^3}{6a^2} + \frac{y^5}{120a^4} - \frac{y^7}{5040a^6}$, &c.

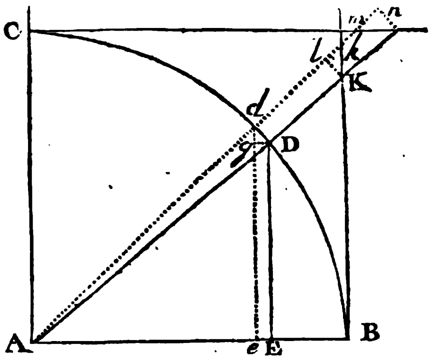
If it were required to extract the Root y out of this Equation, $\dot{y}^2 - x^2\dot{y}^2 + m^2y^2 - m^2a^2 = 0$, (where $\dot{x} = 1$,) we might proceed thus. Becauſe $\dot{y}^2 = \frac{m^2a^2 - m^2y^2}{a^2 - x^2} = m^2$, &c. 'tis $\dot{y} = m$, &c. and $y = * \; mx$, &c. Taking the Fluxions, we ſhall have $2a^2\dot{y}\ddot{y} - 2x\dot{y}^2 - 2x^2\dot{y}\ddot{y} + 2m^2\dot{y}y = 0$, or $a^2\ddot{y} - x\dot{y} - x^2\ddot{y} + m^2y = 0$, or $\ddot{y} = \frac{x\dot{y} - m^2y}{a^2 - x^2} = 0$, &c. Therefore taking the Fluxions again, 'tis $a^2\dddot{y} - \dot{y} - 3x\ddot{y} - x^2\dddot{y} + m^2\dot{y} = 0$, that is, $a^2\dddot{y} + \overline{m^2 - 1} \times \dot{y} - 3x\ddot{y} - x^2\dddot{y} = 0$, or $\dddot{y} = \frac{\overline{1 - m^2} \times \dot{y} + 3x\ddot{y}}{a^2 - x^2}$; and making $\dot{y} = m$, &c. 'tis $\dddot{y} = \frac{m \times \overline{1 - m^2}}{a^2}$, &c. and therefore $\ddot{y} = * \; \frac{m \times \overline{1 - m^2}}{a^2}x$, &c. $\dot{y} = * \; * \; \frac{m \times \overline{1 - m^2}}{2a^2}x^2$, &c. and $y = * \; * \; * \; \frac{m \times \overline{1 - m^2}}{2 \times 3a^2}x^3$, &c. Taking the Fluxions again, 'tis $a^2\ddddot{y} + \overline{m^2 - 4} \times \ddot{y} - 5x\dddot{y} - x^2\ddddot{y} = 0$; and again, $a^2\overset{5}{y} + \overline{m^2 - 9} \times \dddot{y} - 7x\ddddot{y} - x^2\overset{5}{y} = 0$, or $\overset{5}{y} = \frac{\overline{9 - m^2} \times \dddot{y} + 7x\ddddot{y}}{a^2 - x^2} = \frac{m \times \overline{1 - m^2} \times \overline{9 - m^2}}{a^4}$, &c. Therefore $\ddddot{y} = * \; \frac{m \times \overline{1 - m^2} \times \overline{9 - m^2}}{a^4}x$, &c. $\dddot{y} = * \; * \; \frac{m \times \overline{1 - m^2} \times \overline{9 - m^2}}{2a^4}x^2$, &c. $\ddot{y} = * \; * \; * \; \frac{m \times \overline{1 - m^2} \times \overline{9 - m^2}}{2 \times 3a^4}x^3$, &c. $\dot{y} = * \; * \; * \; * \; \frac{m \times \overline{1 - m^2} \times \overline{9 - m^2}}{2 \times 3 \times 4a^4}x^4$, &c. and $y = * \; * \; * \; * \; * \; \frac{m \times \overline{1 - m^2} \times \overline{9 - m^2}}{2 \times 3 \times 4 \times 5a^4}x^5$, &c. And ſo on. Therefore we ſhall have $y = mx + m \times \frac{1 - m^2}{2 \times 3a^2}x^3 + m \times \frac{1 - m^2}{2 \times 3} \times \frac{9 - m^2}{4 \times 5a^4}x^5 + m \times \frac{1 - m^2}{2 \times 3} \times \frac{9 - m^2}{4 \times 5} \times \frac{25 - m^2}{6 \times 7a^6}x^7$, &c.

This Series is equivalent to a Theorem of our Author's, which (in another place) he gives us for Angular Sections. For if x be the Sine of any given Arch, to Radius a; then will y be the Sine of another Arch, which is to the firſt Arch in the given Ratio of m to 1. Here if m be any odd Number, the Series will become finite; and in other caſes it will be a converging Series.

And theſe Examples may be ſufficient to explain this Method of Extraction of Roots; which, tho' it carries its own Demonſtration along with it, yet for greater evidence may be thus farther illuſtrated. In Equations whoſe Roots (for example) may be repreſented by the general Series $y = A + Bx + Cx^2 + Dx^3$, &c. (which by due Reduction may be all Equations whatever,) the firſt Term A of the Root will be a given quantity, or perhaps $= 0$, which is to be known from the circumſtances of the Queſtion, or from the given

Equation, by Methods that have been abundantly explain'd already. Then making $\dot{x}=1$, we ſhall have have $\dot{y}=B+2Cx+3Dx^2$, &c. where B likewiſe is a conſtant quantity, or perhaps $=0$, and repreſents the firſt Term of the Series $\dot{y}$. This therefore is to be derived from the firſt Fluxional Equation, either given or elſe to be found; and then, becauſe it is $\dot{y}=B$, &c. by taking the Fluents it will be $y=*\,Bx$, &c. whence the ſecond Term of the Root will be known. Then becauſe it is $\ddot{y}=2C+6Dx$, &c. or becauſe the conſtant quantity 2C will repreſent the firſt Term of $\ddot{y}$; this is to be derived from the ſecond Fluxional Equation, either given or to be found. And then, becauſe it is $\ddot{y}=2C$, &c. by taking the Fluents it will be $\dot{y}=*\,2Cx$, &c. and again $y=*\,*\,Cx^2$, &c. by which the third Term of the Root will be known. Then becauſe it is $\dddot{y}=6D$, &c. or becauſe the conſtant quantity 6D will repreſent the firſt Term of the Series $\dddot{y}$; this is to be derived from the third Fluxional Equation. And then, becauſe it is $\dddot{y}=6D$, &c. by taking the Fluents it will be $\ddot{y}=*\,6Dx$, &c. $\dot{y}=*\,*\,3Dx^2$, &c. and $y=*\,*\,*\,Dx^3$, &c. by which the fourth Term of the Root will be known. And ſo for all the ſubſequent Terms. And hence it will not be difficult to obſerve the compoſition of the Coefficients in moſt caſes, and thereby diſcover the Law of Continuation, in ſuch Series as are notable and of general uſe.

If you ſhould deſire to know how the foregoing Trigonometrical Equations are derived from the Circle, it may be ſhewn thus: on the Center A, with Radius $AB=a$, let the Quadrantal Arch BC be deſcribed, and draw the Radius AC. Draw the Tangent BK, and through any point of the Circumference D, draw the Secant ADK, meeting the Tangent in K. At any other point *d* of the Circumference, but as near to D as may be, draw the Secant A*dk*, meeting BK in *k*; on Center A, with Radius AK, deſcribe the Arch K*l*, meeting A*k* in *l*. Then ſuppoſing the point *d* continually to approach towards D, till it finally coincides with it, the Trilineum K*lk* will continually approach to a right-lined Triangle, and to ſimilitude with the Triangle ABK: So that when D*d* is a



Moment of the Circumference, it will be $\frac{\mathrm{K}k}{\mathrm{D}d} = \frac{\mathrm{K}k}{\mathrm{K}l} \times \frac{\mathrm{K}l}{\mathrm{D}d} = \frac{\mathrm{AK}}{\mathrm{AB}} \times \frac{\mathrm{AK}}{\mathrm{AB}}$. Make AB $= a$, the Tangent BK $= x$, and the Arch BD $= y$; and inſtead of the Moments Kk and Dd, ſubſtitute the proportional Fluxions $\dot{x}$ and $\dot{y}$, and it will be $\frac{\dot{x}}{\dot{y}} = \frac{a^2 + x^2}{a^2}$, or $a^2\dot{y} + x^2\dot{y} - a^2\dot{x} = 0$.

From D to AB and *de* let fall the Perpendiculars DE and Dg, which Dg meets *de*, parallel to DE, in g. Then the ultimate form of the Trilineum Ddg will be that of a right-lined Triangle ſimilar to DAE. Whence Dd : dg :: AD : AE $= \sqrt{\mathrm{AD}q - \mathrm{DE}q}$. Make AD $= a$, BD $= y$, and DE $= x$; and for the Moments Dd, dg, ſubſtitute their proportional Fluxions $\dot{y}$ and $\dot{x}$, and it will be $\dot{y} : \dot{x} :: a : \sqrt{a^2 - x^2}$. Or $\dot{y}^2 : \dot{x}^2 :: a^2 : a^2 - x^2$, or $a^2\dot{y}^2 - x^2\dot{y}^2 - a^2\dot{x}^2 = 0$.

Hence the Fluxion of an Arch, whoſe right Sine is x, being expreſs'd by $\frac{a\dot{x}}{\sqrt{a^2 - x^2}}$; and likewiſe the Fluxion of an Arch, whoſe right Sine is y, being expreſs'd by $\frac{a\dot{y}}{\sqrt{a^2 - y^2}}$; if theſe Arches are to each other as 1 to m, their Fluxions will be in the ſame proportion, and *vice verſâ*. Therefore $\frac{a\dot{x}}{\sqrt{a^2 - x^2}} : \frac{a\dot{y}}{\sqrt{a^2 - y^2}} :: 1 : m$, or $\frac{m\dot{x}}{\sqrt{a^2 - x^2}} = \frac{\dot{y}}{\sqrt{a^2 - y^2}}$, or $\frac{m^2\dot{x}^2}{a^2 - x^2} = \frac{\dot{y}^2}{a^2 - y^2}$, or putting $\dot{x} = 1$, 'tis $a^2\dot{y}^2 - x^2\dot{y}^2 - m^2a^2 + m^2y^2 = 0$; the ſame Equation as before reſolved.

We might derive other Fluxional Equations, of a like nature with theſe, which would be accommodated to Trigonometrical uſes. As if y were the Circular Arch, and x its verſed Sine, we ſhould have the Equation $2ax\dot{y}^2 - x^2\dot{y}^2 - a^2\dot{x}^2 = 0$. Or if y were the Arch, and x the correſponding Secant, it would be $x^4\dot{y}^2 - a^2x^2\dot{y}^2 - a^4\dot{x}^2 = 0$. Or inſtead of the natural, we might derive Equations for the artificial Sines, Tangents, Secants, &c. But I ſhall leave theſe Diſquiſitions, and many ſuch others that might be propoſed, to exerciſe the Induſtry and Sagacity of the Learner.

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SECT. VI. *An Analytical Appendix, explaining ſome Terms and Expreſſions in the foregoing work.*

BEcauſe mention has been frequently made of *given Equations,* and others *aſſumed ad libitum,* and the like; I ſhall take occaſion from hence, by way of Appendix, to attempt ſome kind of explanation of this Mathematical Language, or of the Terms *given, aſſign'd, aſſumed,* and *required* Quantities or Equations, which may give light to ſome things that may otherwiſe ſeem obſcure, and may remove ſome doubts and ſcruples, which are apt to ariſe in the Mind of a Learner. Now the origin of ſuch kind of Expreſſions in all probability ſeems to be this. The whole affair of purſuing Mathematical Inquiries, or of reſolving Problems, is ſuppoſed (tho' tacitely) to be tranſacted between two Perſons, or Parties, the Propoſer and the Reſolver of the Problem, or (if you pleaſe) between the Maſter (or Inſtructor) and his Scholar. Hence this, and ſuch like Phraſes, *datam rectam, vel datum angulum, in imperatâ ratione ſecare.* As Examples inſtruct better than Precepts, or perhaps when both are join'd together they inſtruct beſt, the Maſter is ſuppos'd to propoſe a Queſtion or Problem to his Scholar, and to chuſe ſuch Terms and Conditions as he thinks fit; and the Scholar is obliged to ſolve the Problem with thoſe limitations and reſtrictions, with thoſe Terms and Conditions, and no other. Indeed it is required on the part of the Maſter, that the Conditions he propoſes may be conſiſtent with one another; for if they involve any inconſiſtency or contradiction, the Problem will be unfair, or will become abſurd and impoſſible, as the Solution will afterwards diſcover. Now theſe Conditions, theſe Points, Lines, Angles, Numbers, Equations, *&c.* that at firſt enter the ſtate of the Queſtion, or are ſuppoſed to be choſen or given by the Maſter, are the *data* of the Problem, and the Anſwers he expects to receive are the *quæſita.* As it may ſometimes happen, that the *data* may be more than are neceſſary for determining the Queſtion, and ſo perhaps may interfere with one another, and the Problem (as now propoſed) may become impoſſible; ſo they may be fewer than are neceſſary, and the Problem thence will be indetermin'd, and may require other Conditions to be given, in order to a compleat Determination, or perfectly to fulfil the *quæſita.* In this caſe the Scholar is to ſupply what is wanting, and at his diſcretion may *aſſume* ſuch and ſo many other Terms and Conditions, Equations and Limitations, as he finds will be neceſſary to his purpoſe, and will beſt conduce to the ſimpleſt, the eaſieſt, and neateſt Solution that may be had, and yet in the moſt general manner. For it is convenient the Problem ſhould be propoſed as particular as may be, the better to fix the Imagination; and yet the Solution ſhould be made as general as poſſible, that it may be the more inſtructive, and extend to all caſes of a like nature.

Indeed the word *datum* is often uſed in a ſenſe which is ſomething different from this, but which ultimately centers in it. As that is call'd a *datum*, when one quantity is not immediately given, but however is neceſſarily infer'd from another, which other perhaps is neceſſarily infer'd from a third, and ſo on in a continued Series, till it is neceſſarily infer'd from a quantity, which is known or given in the ſenſe before explain'd. This is the Notion of *Euclid*'s *data*, and other Analytical Argumentations of that kind. Again, that is often call'd a *given quantity*, which always remains conſtant and invariable, while other quantities or circumſtances vary; becauſe ſuch as theſe only can be the given quantities in a Problem, when taken in the foregoing ſenſe.

To make all this the more ſenſible and intelligible, I ſhall have recourſe to a few practical inſtances, by way of Dialogue, (which was the old didactic method,) between Maſter and Scholar; and this only in the common Algebra or Analyticks, in which I ſhall borrow my Examples from our Author's admirable Treatiſe of Univerſal Arithmetick. The chief artifice of this manner of Solution will conſiſt in this, that as faſt as the Maſter propoſes the Conditions of his Queſtion, the Scholar applies thoſe Conditions to uſe, argues from them Analytically, makes all the neceſſary deductions, and derives ſuch conſequences from them, in the ſame order they are propoſed, as he apprehends will be moſt ſubſervient to the Solution. And he that can do this in all caſes, after the ſureſt, ſimpleſt, and readieſt manner, will be the beſt *ex-tempore* Mathematician. But this method will be beſt explain'd from the following Examples.

I. M. *A Gentleman being willing to diſtribute Alms* --- S. Let the Sum he intended to diſtribute be repreſented by x. M. *Among ſome poor people.* S. Let the number of poor be y, then $\frac{x}{y}$ would have been the ſhare of each. M. *He wanted 3 ſhillings* --- S. Make $3 = a$, for the ſake of univerſality, and let the pecuniary Unit be one Shilling; then the Sum to be diſtributed would have been $x + a$, and the ſhare of each would have been $\frac{x+a}{y}$. M. *So that each might receive* 5 *ſhillings.* S. Make $5=b$, then $\frac{x+a}{y}=b$, whence $x=by-a$. M. *Therefore he gave every one* 4 *ſhillings.* S. Make $4=c$, then the Money diſtributed will be cy. M. *And he has* 10 *ſhillings remaining.* S. Make $10=d$, then $cy+d$ was the Money he intended at firſt to diſtribute; or $cy+d=(x=)\ by-a$, or $y=\frac{a+d}{b-c}$. M. *What was the number of poor people?* S. The number was $y=\frac{a+d}{b-c}=\frac{3+10}{5-4}=13$. M. *And how much Alms did he at firſt intend to diſtribute?* S. He had at firſt $x=by-a$ $=5\times13-3=62$ ſhillings. M. *How do you prove your Solution?* S. His Money was at firſt 62 ſhillings, and the number of poor people was 13. But if his Money had been $62+3=65=13\times5$ ſhillings, then each poor perſon might have received 5 ſhillings. But as he gives to each 4 ſhillings, that will be $13\times4=52$ ſhillings diſtributed in all, which will leave him a Remainder of $62-52$ $=10$ ſhillings.

II. M. *A young Merchant, at his firſt entrance upon buſineſs, began the World with a certain Sum of Money.* S. Let that Sum be x, the pecuniary Unit being one Pound. M. *Out of which, to maintain himſelf the firſt year, he expended* 100 *pounds.* S. Make the given number $100=a$; then he had to trade with $x-a$. M. *He traded with the reſt, and at the end of the year had improved it by a third part.* S. For univerſality-ſake I will aſſume the general number n, and will make $\frac{1}{3}=n-1$, (or $n=\frac{4}{3}$;) then the Improvement was $\overline{n-1}\times\overline{x-a}=nx-na-x+a$, and the Trading-ſtock and Improvement together, at the end of the firſt year, was $nx-na$. M. *He did the ſame thing the ſecond year.* S. That is, his whole Stock being now $nx-na$, deducting a, his Expences for this year, he would have $nx-na-a$ for a Trading-ſtock, and $\overline{n-1}\times\overline{nx-na-a}$, or $n^2x-n^2a-nx+a$ for this year's Improvement, which together make n^2x-n^2a-na for his Eſtate at the end of the ſecond year. M. *As alſo the third year.* S. His whole Stock being now n^2x-n^2a-na, taking out his Expences for the third year, his Trading-ſtock will be $n^2x-n^2a-na-a$, and the Improvement this year will be $\overline{n-1}\times\overline{n^2x-n^2a-na-a}$, or $n^3x-n^3a-n^2x+a$, and the Stock and Improvement together, or his whole Eſtate at the end of the third year will be n^3x-n^3a $-n^2a-na$, or in a better form $n^3x+\frac{n^3-1}{1-n}na$. In like manner if he proceeded thus the fourth year, his Eſtate being now $n^3x - n^3a - n^2a - na$, taking out this year's Expence, his Trading-ſtock will be $n^3x - n^3a - n^2a - na - a$, and this year's Improvement is $\overline{n-1} \times \overline{n^3x - n^3a - n^2a - na - a}$, or $n^4x - n^4a - n^3x + a$, which added to his Trading-ſtock will be $n^4x - n^4a - n^3a - n^2a - na$, or $n^4x + \frac{n^4-1}{1-n}na$, for his Eſtate at the end of the fourth year. And ſo, by Induction, his Eſtate will be found $n^5x + \frac{n^5-1}{1-n}na$ at the end of the fifth year. And univerſally, if I aſſume the general Number m, his Eſtate will be $n^mx + \frac{n^m-1}{1-n}na$ at the end of any number of years denoted by m. M. *But he made his Eſtate double to what it was at firſt.* S. Make $2 = b$, then $n^mx + \frac{n^m-1}{1-n}na = bx$, or $x = \frac{n^m-1}{\overline{n-1} \times n^m - b}na$. M. *At the end of* 3 *years.* S. Then $m = 3$, $a = 100$, $b = 2$, $n = \frac{4}{3}$, and therefore $x = \frac{\frac{4^3}{3^3} - 1}{\frac{1}{3} \times \frac{4^3}{3^3} - 2} \times \frac{4}{3} \times 100 = \frac{4^3 - 3^3}{4^3 - 2 \times 3^3} \times 400 = \frac{64 - 27}{64 - 54} \times 400 = \frac{37}{10} \times 400 = 1480$. M. *What was his Eſtate at firſt?* S. It was 1480 pounds.

III. M. *Two Bodies* A *and* B *are at a given diſtance from each other.* S. As their diſtance is ſaid to be given, though it is not ſo actually, I may therefore aſſume it. Let the initial diſtance of the Bodies be $59 = e$, and let the Linear Unit be one Mile. M. *And move equably towards one another.* S. Let x repreſent the whole ſpace deſcribed by A before they meet; then will $e - x$ be the whole ſpace deſcribed by B. M. *With given Velocities.* S. I will aſſume the Velocity of A to be ſuch, that it will move $7 = c$ Miles in $2 = f$ Hours, the Unit of Time being one Hour. Then becauſe it is $c : f :: x : \frac{fx}{c}$, A will move his whole ſpace x in the time $\frac{fx}{c}$. Alſo I will aſſume the Velocity of B to be ſuch, that it will move $8 = d$ Miles in $3 = g$ Hours. Then becauſe it is $d : g :: e - x : \frac{e-x}{d}g$, B will move his whole ſpace $e - x$ in the time $\frac{e-x}{d}g$. M. *But* A *moves a given time* - - - S. Let that time be $1 = h$ Hour. M. *Before* B *begins to move.* S. Then A's time is equal to B's time added to the time h, or $\frac{fx}{c} = \frac{e-x}{d}g + h$.

M.

M. *Where will they meet, or what will be the ſpace that each will have deſcribed?* S. From this Equation we ſhall have $x = \frac{eg + dh}{df + cg} c = \frac{59 \times 3 + 8 \times 1}{8 \times 2 + 7 \times 3} \times 7 = \frac{185}{37} \times 7 = 5 \times 7 = 35$ Miles, which will be the whole ſpace deſcribed by *A*. Then $e - x = 59 - 35 = 24$ Miles will be the whole ſpace deſcribed by *B*.

IV. M. *If* 12 *Oxen can be maintained by the Paſture of* $3\frac{1}{3}$ *Acres of Meadow-ground for* 4 *weeks*, S. Make $12 = a$, $3\frac{1}{3} = b$, $4 = c$; then aſſuming the general Numbers e, f, b, to be determin'd afterwards as occaſion ſhall require, we ſhall have by analogy

		Oxen		Paſture		Time
If	Directly	a	require	b	during	c
Then		ae		be		c
Alſo		$\frac{ae}{b}$		e		c
And	Reciprocally	$\frac{ace}{b}$		e		1
Alſo		$\frac{ace}{bf}$		e		f
Alſo		$\frac{ace}{bb}$		e		b

M. *And if, becauſe of the continual growth of the Graſs after the four weeks, it be found that* 21 *Oxen can be maintain'd by the paſture of* 10 *ſuch Acres for* 9 *weeks*, S. Make $21 = d$, $e = 10$, $f = 9$; then becauſe on this ſuppoſition, the Oxen d require the paſture e during the time f; and in the former caſe the Oxen $\frac{ace}{bf}$ required the ſame paſture during the ſame time: Therefore the growth of the Graſs of the quantity of paſture e, (commencing after 4 or c weeks, and continuing to the end of the Time f, or during the whole time $f - c$,) is ſuch, as alone was ſufficient to maintain the difference of the Oxen, or the number $d - \frac{ace}{bf}$, for the whole time f. Then reciprocally that growth would be ſufficient to maintain the number of Oxen $df - \frac{ace}{b}$ for the time 1, or the number of Oxen $\frac{df}{b} - \frac{ace}{bb}$ for the time b. And becauſe this growth will be proportional to the time, and will maintain a greater number of Oxen in proportion as the time is greater; we ſhall have

Time

$$\begin{array}{cccc} \text{Time} & \text{Oxen} & \text{Time} & \text{Oxen} \\ f - c \,. & \frac{df}{h} - \frac{ace}{bh} :: & h - c \,. & \frac{h-c}{f-c} \text{ into } \frac{df}{h} - \frac{ace}{bh}, \end{array}$$

which will be the number of Oxen that may be maintain'd by the growth only of the paſture *e*, during the whole time *h*. But it was found before, that without this growth of the Graſs, the Oxen $\frac{ace}{bh}$ might be maintain'd by the paſture *e* for the time *h*. Therefore theſe two together, or $\frac{ace}{bh} + \frac{h-c}{f-c} \times \frac{bdf - ace}{bh}$, will be the number of Oxen that may be maintain'd by the paſture *e*, and its growth together, during the time *h*. M. *How many Oxen may be maintain'd by* 24 *Acres of ſuch paſture for* 18 *weeks?* *S.* Suppoſe *x* to be that number of Oxen, and make $24 = g$, and $h = 18$. Then by analogy

$$\begin{array}{lccc} & \text{Oxen} & & \text{Paſture} \\ \begin{matrix} \text{If} \\ \text{Then} \\ \text{And} \end{matrix} & \left\{ \begin{matrix} x \\ ex \\ \frac{ex}{g} \end{matrix} \right\} & \text{require} & \left\{ \begin{matrix} g \\ eg \\ e \end{matrix} \right\} \text{ during the time } h. \end{array}$$

And conſequently $\frac{ex}{g} = \frac{ace}{bh} + \frac{h-c}{f-c} \times \frac{bdf - ace}{bh}$, or $x = \frac{acg}{bh} + \frac{h-c}{f-c} \times \overline{\frac{dfg}{eh} - \frac{acg}{bh}} = \frac{ac}{b} + \frac{h-c}{f-c} \times \overline{\frac{df}{e} - \frac{ac}{b}}$ into $\frac{g}{h} = \frac{12 \times 4}{3\frac{1}{3}} + \frac{18-4}{9-4} \times \overline{\frac{21 \times 9}{10} - \frac{12 \times 4}{3\frac{1}{3}}}$ into $\frac{24}{18} = 36$.

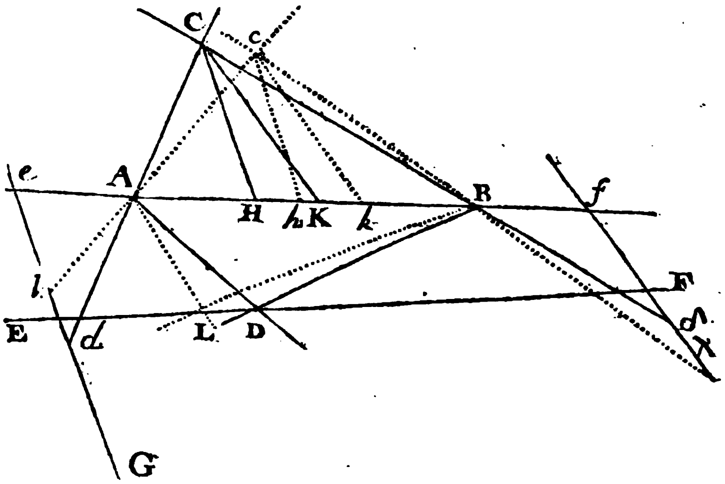
V. M. *If I have an Annuity* - - - *S.* Let *x* be the preſent value of 1 pound to be received 1 year hence, then (by analogy) x^2 will be the preſent value of 1 pound to be received 2 years hence, &c. and in general, x^m will be the preſent value of 1 pound to be received *m* years hence. Therefore, in the caſe of an Annuity, the Series $x + x^2 + x^3 + x^4$, &c. to be continued to ſo many Terms as there are Units in *m*, will be the preſent value of the whole Annuity of 1 pound, to be continued for *m* years. But becauſe $\frac{x - x^{m+1}}{1-x} = x + x^2 + x^3 + x^4$, &c. continued to ſo many Terms as there are Units in *m*, (as may appear by Diviſion;) therefore $\frac{x - x^{m+1}}{1-x}$ will repreſent the Amount of an Annuity of 1 pound, to be continued for *m* years. M. *Of* *Pounds.* *S.* Make

$= a,$

$= a$, then the Amount of this Annuity for m years will be $\frac{x - x^{m+1}}{1-x} a$. M. *To be continued for 5 years ſucceſſively.* S. Then $m = 5$. M. *Which I ſell for   pounds in ready Money.* S. Make $= c$, then $\frac{x - x^{m+1}}{1-x} a = c$; or $x^{m+1} - \overline{\frac{c}{a} + 1} \times x + \frac{c}{a} = 0$. In any particular caſe the value of x may be found by the Reſolution of this affected Equation. M. *What Intereſt am I allow'd* per centum per annum? S. Make $100 = b$; then becauſe x is the preſent value of 1 pound to be received 1 year hence, or (which is the ſame thing) becauſe the preſent Money x, if put out to uſe, in 1 year will produce 1 pound; the Intereſt alone of 1 pound for 1 year will be $1 - x$, and therefore the Intereſt of 100 (or b) pounds for 1 year will be $b - bx$, which will be known when x is known.

And this might be ſufficient to ſhew the conveniency of this Method; but I ſhall farther illuſtrate it by one Geometrical Problem, which ſhall be our Author's LVII.

VI. M. *In the right Line* AB *I give you the two points* A *and* B. S. Then their diſtance $AB = m$ is given alſo. M. *As likewiſe the two points* C *and* D *out of the Line* AB. S. Then conſequently the figure ACBD is given in magnitude and ſpecie; and producing CA and CB towards d and δ, I can take $Ad = AD$, and $B\delta = BD$. M. *Alſo I give you the indefinite right Line* EF *in poſition, paſſing thro' the given point* D. S. Then the Angles ADE and BDF are given, to which (producing AB both ways, if need be, to e and f,) I can make the Angles Ade and $B\delta f$ equal reſpectively, and that will determine the points e and f, or the Lines $Ae = a$, and $Bf = c$. And becauſe de and δf are thereby known, I can continue de to G, ſo that $dG = \delta f$, and make the given line $eG = b$. Likewiſe I can draw CH and



and CK parallel to *ed* and *f*δ reſpectively, meeting AB in H and K; and becauſe the Triangle CHK will be given in magnitude and ſpecie, I will make CK $=d$, CH $=e$, and HK $=f$. M. *Now let the given Angles* CAD *and* CBD *be conceived to revolve about the given points or Poles* A *and* B. *S.* Then the Lines AD and CA*d* will move into another ſituation AL and *c*A*l*, ſo as that the Angles DAL, *d*A*l*, and CA*c* will be equal. Alſo the Lines BD and CBδ will obtain a new ſituation BL and *c*Bλ, ſo as that the Angles DBL, δBλ and CB*c* will be equal. M. *And let* D, *the Interſection of the Lines* AD *and* BD, *always move in the right Line* EF. *S.* Then the new point of Interſection L is in EF; then the Triangles DAL and *d*A*l*, as alſo DBL and δBλ, are equal and ſimilar; then $dl =$ DL $= \delta\lambda$, and therefore G$l = f\lambda$. M. *What will be the nature of the Curve deſcribed by the other point of Interſection* C? *S.* From the new point of Interſection *c* to AB, I will draw the Lines *cb* and *ck*, parallel to CH and CK reſpectively. Then will the Triangle *cbk* be given in ſpecie, though not in magnitude, for it will be ſimilar to CHK. Alſo the Triangle B*ck* will be ſimilar to Bλ*f*. And the indefinite Line B$k = x$ may be aſſumed for an Abſciſs, and $ck = y$ may be the correſponding Ordinate to the Curve C*c*. Then becauſe it is B*k* $(x) : ck\ (y) ::$ B*f* $(c) : f\lambda = \frac{cy}{x} =$ G*l*. Subtract this from G$e = b$; and there will remain $le = b - \frac{cy}{x}$. Then becauſe of the ſimilar Triangles *cbk* and CHK, it will be CK (d) : CH $(e) :: ck\ (y) : cb = \frac{ey}{d}$. And CK (d) : HK $(f) :: ck\ (y) : bk = \frac{fy}{d}$. Therefore A$b =$ AB $-$ B$k - bk = m - x - \frac{fy}{d}$. But it is A$b$ $\left(m - x - \frac{fy}{d}\right) : cb \left(\frac{ey}{d}\right) ::$ Ae $(a) : le \left(b - \frac{cy}{x}\right)$. Therefore $\overline{m - x - \frac{fy}{x}} \times \overline{b - \frac{ey}{d}} = \frac{aey}{d}$, or $fcy^2 + \overline{dc - ae - bf} \times xy - dcmy - bdx^2 + bdmx = 0$. In which Equation, becauſe the indeterminate quantities x and y ariſe only to two Dimenſions, it ſhews that the Curve deſcribed by the point C is a Conic Section.

M. *You have therefore ſolved the Problem in general, but you ſhould now apply your Solution to the ſeveral ſpecies of Conic Sections in particular.* *S.* That may eaſily be done in the following manner: Make $\frac{ae + bf - cd}{c} = 2p$, and then the foregoing Equation will become $fcy^2 - 2pcxy - dcmy - bdx^2 + bdmx = 0$, and by extracting the Square-root it will be $y = \frac{p}{f}x + \frac{dm}{2f} \pm \sqrt{\overline{\frac{pp}{ff} + \frac{bd}{fc}} \times x^2 + \overline{\frac{pdm}{ff} - \frac{bdm}{fc}} \times x + \frac{d^2m^2}{4ff}}$. Now here it is plain, that if the Term $\overline{\frac{pp}{ff} + \frac{bd}{fc}} \times x^2$ were abſent, or if $\frac{pp}{ff} + \frac{bd}{fc} = 0$, or $\frac{pp}{ff} = -\frac{bd}{fc}$; that is, if the quantity $\frac{bd}{fc}$ (changing its ſign) ſhould be equal to $\frac{pp}{ff}$, then the Curve would be a Parabola. But if the ſame Term were preſent, and equal to ſome affirmative quantity, that is, if $\frac{pp}{ff} + \frac{bd}{fc}$ be affirmative, (which will always be when $\frac{bd}{fc}$ is affirmative, or if it be negative and leſs than $\frac{pp}{ff}$,) the Curve will be an Hyperbola. Laſtly, if the ſame Term were preſent and negative, (which can only be when $\frac{bd}{fc}$ is negative, and greater than $\frac{pp}{ff}$,) the Curve will be an Ellipſis or a Circle.

I ſhould make an apology to the Reader, for this Digreſſion from the Method of Fluxions, if I did not hope it might contribute to his entertainment at leaſt, if not to his improvement. And I am fully convinced by experience, that whoever ſhall go through the reſt of our Author's curious Problems, in the ſame manner, (wherein, according to his uſual brevity, he has left many things to be ſupply'd by the ſagacity of his Reader,) or ſuch other Queſtions and Mathematical Diſquiſitions, whether Arithmetical, Algebraical, Geometrical, &c. as may eaſily be collected from Books treating on theſe Subjects; I ſay, whoever ſhall do this after the foregoing manner, will find it a very agreeable as well as profitable exerciſe: As being the proper means to acquire a habit of Inveſtigation, or of arguing ſurely, methodically, and Analytical[illegible], even in other Sciences as well as ſuch as are purely Mathematical; which is the great end to be aim'd at by theſe Studies.

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SECT. VII. *The Concluſion; containing a ſhort recapitulation or review of the whole.*

WE are now arrived at a period, which may properly enough be call'd *the concluſion of the Method of Fluxions and Infinite Series*; for the deſign of this Method is to teach the nature of Series in general, and of Fluxions and Fluents, what they are, how they are derived, and what Operations they may undergo; which deſign (I think) may now be ſaid to be accompliſh'd. As to the application of this Method, and the uſes of theſe Operations, which is all that now remains, we ſhall find them inſiſted on at large by the Author in the curious Geometrical Problems that follow. For the whole that can be done, either by Series or by Fluxions, may eaſily be reduced to the Reſolution of Equations, either Algebraical or Fluxional, as it has been already deliver'd, and will be farther apply'd and purſued in the ſequel. I have continued my Annotations in a like manner upon that part of the Work, and intended to have added them here; but finding the matter to grow ſo faſt under my hands, and ſeeing how impoſſible it was to do it juſtice within ſuch narrow limits, and alſo perceiving this work was already grown to a competent ſize; I reſolved to lay it before the Mathematical Reader unfiniſh'd as it is, reſerving the completion of it to a future opportunity, if I ſhall find my preſent attempts to prove acceptable. Therefore all that remains to be done here is this, to make a kind of review of what has been hitherto deliver'd, and to give a ſummary account of it, in order to acquit myſelf of a Promiſe I made in the Preface. And having there done this already, as to the Author's part of the work, I ſhall now only make a ſhort recapitulation of what is contain'd in my own Comment upon it.

And firſt in my Annotations upon what I call the Introduction, or the Reſolution of Equations by infinite Series, I have amply purſued a uſeful hint given us by the Author, that Arithmetick and Algebra are but one and the ſame Science, and bear a ſtrict analogy to each other, both in their Notation and Operations; the firſt computing after a definite and particular manner, the latter after a general and indefinite manner: So that both together compoſe but one uniform Science of Computation. For as in common Arithmetick we reckon by the Root *Ten*, and the ſeveral Powers of that Root; ſo in Algebra, or Analyticks, when the Terms are orderly diſpos'd as is preſcribed, we reckon by any other Root and its Powers, or we may take any general Number for the Root of our Arithmetical Scale, by which to expreſs and compute any Numbers required. And as in common Arithmetick we approximate continually to the truth, by admitting Decimal Parts *in infinitum*, or by the uſe of Decimal Fractions, which are compoſed of the reciprocal Powers of the Root *Ten*; ſo in our Author's improved Algebra, or in the Method of infinite converging Series, we may continually approximate to the Number or Quantity required, by an orderly ſucceſſion of Fractions, which are compoſed of the reciprocal Powers of any Root in general. And the known Operations in common Arithmetick, having a due regard to Analogy, will generally afford us proper patterns and ſpecimens, for performing the like Operations in this Univerſal Arithmetick.

Hence I proceed to make ſome Inquiries into the nature and formation of infinite Series in general, and particularly into their two principal circumſtances of Convergency and Divergency; wherein I attempt to ſhew, that in all ſuch Series, whether converging or diverging, there is always a Supplement, which if not expreſs'd is however to be underſtood; which Supplement, when it can be aſcertained and admitted, will render the Series finite, perfect, and accurate. That in diverging Series this Supplement muſt indiſpenſably be admitted and exhibited, or otherwiſe the Concluſion will be imperfect and erroneous. But in converging Series this Supplement may be neglected, becauſe it continually diminiſhes with the Terms of the Series, and finally becomes leſs than any aſſignable quantity. And hence ariſes the benefit and conveniency of infinite converging Series; that whereas that Supplement is commonly ſo implicated and entangled with the Terms of the Series, as often to be impoſſible to be extricated and exhibited; in converging Series it may ſafely be neglected, and yet we ſhall continually approximate to the quantity required. And of this I produce a variety of Inſtances, in numerical and other Series.

I then go on to ſhew the Operations, by which infinite Series are either produced, or which, when produced, they may occaſionally undergo. As firſt when ſimple ſpecious Equations, or pure Powers, are to be reſolved into ſuch Series, whether by Diviſion, or by Extraction of Roots; where I take notice of the uſe of the afore-mention'd Supplement, by which Series may be render'd finite, that is, may be compared with other quantities, which are conſider'd as given. I then deduce ſeveral uſeful Theorems, or other Artifices, for the more expeditious Multiplication, Diviſion, Involution, and Evolution of infinite Series, by which they may be eaſily and readily managed in all caſes. Then I ſhew the uſe of theſe in pure Equations, or Extractions; from whence I take occaſion to introduce a new praxis of Reſolution, which I believe will be found to be very eaſy, natural, and general, and which is afterwards apply'd to all ſpecies of Equations.

Then I go on with our Author to the *Exegeſis numeroſa*, or to the Solution of affected Equations in Numbers; where we ſhall find his Method to be the ſame that has been publiſh'd more than once in other of his pieces, to be very ſhort, neat, and elegant, and was a great Improvement at the time of its firſt publication. This Method is here farther explain'd, and upon the ſame Principles a general Theorem is form'd, and diſtributed into ſeveral ſubordinate Caſes, by which the Root of any Numerical Equation, whether pure or affected, may be computed with great exactneſs and facility.

From Numeral we paſs on to the Reſolution of Literal or Specious affected Equations by infinite Series; in which the firſt and chief difficulty to be overcome, conſiſts in determining the forms of the ſeveral Series that will ariſe, and in finding their initial Approximations. Theſe circumſtances will depend upon ſuch Powers of the Relate and Correlate Quantities, with their Coefficients, as may happen to be found promiſcuouſly in the given Equation. Therefore the Terms of this Equation are to be diſpoſed *in longum & in latum*, or at leaſt the Indices of thoſe Powers, according to a combined Arithmetical Progreſſion *in plano*, as is there explain'd; or according to our Author's ingenious Artifice of the Parallelogram and Ruler, the reaſon and foundation of which are here fully laid open. This will determine all the caſes of exterior Terms, together with the Progreſſions of the Indices; and therefore all the forms of the ſeveral Series that may be derived for the Root, as alſo their initial Coefficients, Terms, or Approximations.

We then farther proſecute the Reſolution of Specious Equations, by diverſe Methods of Analyſis; or we give a great variety of Proceſſes, by which the Series for the Roots are eaſily produced to any number of Terms required. Theſe Proceſſes are generally very ſimple, and depend chiefly upon the Theorems before deliver'd, for finding the Terms of any Power or Root of an infinite Series. And the whole is illuſtrated and exemplify'd by a great variety of Inſtances, which are chiefly thoſe of our Author.

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The Method of infinite Series being thus ſufficiently diſcuſs'd, we make a Tranſition to the Method of Fluxions, wherein the nature and foundation of that Method is explain'd at large. And ſome general Obſervations are made, chiefly from the Science of Rational Mechanicks, by which the whole Method is divided and diſtinguiſh'd into its two grand Branches or Problems, which are the Direct and Inverſe Methods of Fluxions. And ſome preparatory Notations are deliver'd and explain'd, which equally concern both theſe Methods.

I then proceed with my Annotations upon the Author's firſt Problem, or the Relation of the flowing Quantities being given, to determine the Relation of their Fluxions. I treat here concerning Fluxions of the firſt order, and the method of deducing their Equations in all caſes. I explain our Author's way of taking the Fluxions of any given Equation, which is much more general and ſcientifick than that which is uſually follow'd, and extends to all the varieties of Solutions. This is alſo apply'd to Equations involving ſeveral flowing Quantities, by which means it likewiſe comprehends thoſe caſes, in which either compound, irrational, or mechanical Quantities may be included. But the Demonſtration of Fluxions, and of the Method of taking them, is the chief thing to be conſider'd here; which I have endeavour'd to make as clear, explicite, and ſatisfactory as I was able, and to remove the difficulties and objections that have been raiſed againſt it: But with what ſucceſs I muſt leave to the judgment of others.

I then treat concerning Fluxions of ſuperior orders, and give the Method of deriving their Equations, with its Demonſtration. For tho' our Author, in this Treatiſe, does not expreſſly mention theſe orders of Fluxions, yet he has ſometimes recourſe to them, tho' tacitely and indirectly. I have here ſhewn, that they are a neceſſary reſult from the nature and notion of firſt Fluxions; and that all theſe ſeveral orders differ from each other, not abſolutely and eſſentially, but only relatively and by way of compariſon. And this I prove as well from Geometry as from Analyticks; and I actually exhibit and make ſenſible theſe ſeveral orders of Fluxions.

But more eſpecially in what I call the Geometrical and Mechanical Elements of Fluxions, I lay open a general Method, by the help of Curve-lines and their Tangents, to repreſent and exhibit Fluxions and Fluents in all caſes, with all their concomitant Symptoms and

Affections, after a plain and familiar manner, and that even to ocular view and inſpection. And thus I make them the Objects of Senſe, by which not only their exiſtence is proved beyond all poſſible contradiction, but alſo the Method of deriving them is at the ſame time fully evinced, verified, and illuſtrated.

Then follow my Annotations upon our Author's ſecond Problem, or the Relation of the Fluxions being given, to determine the Relation of the flowing Quantities or Fluents; which is the ſame thing as the Inverſe Method of Fluxions. And firſt I explain (what our Author calls) a particular Solution of this Problem, becauſe it cannot be generally apply'd, but takes place only in ſuch Fluxional Equations as have been, or at leaſt might have been, previouſly derived from ſome finite Algebraical or Fluential Equations. Whereas the Fluxional Equations that uſually occur, and whoſe Fluents or Roots are required, are commonly ſuch as, by reaſon of Terms either redundant or deficient, cannot be reſolved by this particular Solution; but muſt be refer'd to the following general Solution, which is here diſtributed into theſe three Caſes of Equations.

The firſt Caſe of Equations is, when the Ratio of the Fluxions of the Relate and Correlate Quantities, (which Terms are here explain'd,) can be expreſs'd by the Terms of the Correlate Quantity alone; in which Caſe the Root will be obtain'd by an eaſy proceſs: In finite Terms, when it may be done, or at leaſt by an infinite Series. And here a uſeful Rule is explain'd, by which an infinite Expreſſion may be always avoided in the Concluſion, which otherwiſe would often occur, and render the Solution inexplicable.

The ſecond Caſe of Equations comprehends ſuch Fluxional Equations, wherein the Powers of the Relate and Correlate Quantities, with their Fluxions, are any how involved. Tho' this Caſe is much more operoſe than the former, yet it is ſolved by a variety of eaſy and ſimple Analyſes, (more ſimple and expeditious, I think, than thoſe of our Author,) and is illuſtrated by a numerous collection of Examples.

The third and laſt Caſe of Fluxional Equations is, when there are more than two Fluents and their Fluxions involved; which Caſe, without much trouble, is reduced to the two former. But here are alſo explain'd ſome other matters, farther to illuſtrate this Doctrine; as the Author's Demonſtration of the Inverſe Method of Fluxions, the Rationale of the Tranſmutation of the Origin of Fluents to other

places

places at pleaſure, the way of finding the contemporaneous Increments of Fluents, and ſuch like.

Then to conclude the Method of Fluxions, a very convenient and general Method is propoſed and explain'd, for the Reſolution of all kinds of Equations, Algebraical or Fluxional, by having recourſe to ſuperior orders of Fluxions. This Method indeed is not contain'd in our Author's preſent Work, but is contrived in purſuance of a notable hint he gives us, in another part of his Writings. And this Method is exemplify'd by ſeveral curious and uſeful Problems.

Laſtly, by way of Supplement or Appendix, ſome Terms in the Mathematical Language are farther explain'd, which frequently occur in the foregoing work, and which it is very neceſſary to apprehend rightly. And a ſort of Analytical Praxis is adjoin'd to this Explanation, to make it the more plain and intelligible; in which is exhibited a more direct and methodical way of reſolving ſuch Algebraical or Geometrical Problems as are uſually propoſed; or an attempt is made, to teach us to argue more cloſely, diſtinctly, and Analytically.

And this is chiefly the ſubſtance of my Comment upon this part of our Author's work, in which my conduct has always been, to endeavour to digeſt and explain every thing in the moſt direct and natural order, and to derive it from the moſt immediate and genuine Principles. I have always put myſelf in the place of a Learner, and have endeavour'd to make ſuch Explanations, or to form this into ſuch an Inſtitution of Fluxions and infinite Series, as I imagined would have been uſeful and acceptable to myſelf, at the time when I firſt enter'd upon theſe Speculations. Matters of a trite and eaſy nature I have paſs'd over with a ſlight animadverſion: But in things of more novelty, or greater difficulty, I have always thought myſelf obliged to be more copious and explicite; and am conſcious to myſelf, that I have every where proceeded *cum ſincero animo docendi*. Wherever I have fallen ſhort of this deſign, it ſhould not be imputed to any want of care or good intentions, but rather to the want of ſkill, or to the abſtruſe nature of the ſubject. I ſhall be glad to ſee my defects ſupply'd by abler hands, and ſhall always be willing and thankful to be better inſtructed.

What perhaps will give the greateſt difficulty, and may furniſh moſt matter of objection, as I apprehend, will be the Explanations before given, of *Moments, vaniſhing quantities, infinitely little quantities,*

tities, and the like, which our Author makes uſe of in this Treatiſe, and elſewhere, for deducing and demonſtrating his Method of Fluxions. I ſhall therefore here add a word or two to my foregoing Explanations, in hopes farther to clear up this matter. And this ſeems to be the more neceſſary, becauſe many difficulties have been already ſtarted about the abſtract nature of theſe quantities, and by what name they ought to be call'd. It has even been pretended, that they are utterly impoſſible, inconceiveable, and unintelligible, and it may therefore be thought to follow, that the Concluſions derived by their means muſt be precarious at leaſt, if not erroneous and impoſſible.

Now to remove this difficulty it ſhould be obſerved, that the only Symbol made uſe of by our Author to denote theſe quantities, is the letter *o*, either by itſelf, or affected by ſome Coefficient. But this Symbol *o* at firſt repreſents a finite and ordinary quantity, which muſt be underſtood to diminiſh continually, and as it were by local Motion; till after ſome certain time it is quite exhauſted, and terminates in mere nothing. This is ſurely a very intelligible Notion. But to go on. In its approach towards nothing, and juſt before it becomes abſolute nothing, or is quite exhauſted, it muſt neceſſarily paſs through vaniſhing quantities of all proportions. For it cannot paſs from being an aſſignable quantity to nothing at once; that were to proceed *per ſaltum*, and not continually, which is contrary to the Suppoſition. While it is an aſſignable quantity, tho' ever ſo little, it is not yet the exact truth, in geometrical rigor, but only an Approximation to it; and to be accurately true, it muſt be leſs than any aſſignable quantity whatſoever, that is, it muſt be a vaniſhing quantity. Therefore the Conception of a Moment, or vaniſhing quantity, muſt be admitted as a rational Notion.

But it has been pretended, that the Mind cannot conceive quantity to be ſo far diminiſh'd, and ſuch quantities as theſe are repreſented as impoſſible. Now I cannot perceive, even if this impoſſibility were granted, that the Argumentation would be at all affected by it, or that the Concluſions would be the leſs certain. The impoſſibility of Conception may ariſe from the narrowneſs and imperfection of our Faculties, and not from any inconſiſtency in the nature of the thing. So that we need not be very ſolicitious about the poſitive nature of theſe quantities, which are ſo volatile, ſubtile, and fugitive, as to eſcape our Imagination; nor need we be much in pain, by what name they are to be call'd; but we may confine ourſelves wholly to the uſe of them, and to diſcover their properties. They are not introduced for their own ſakes, but only as ſo many intermediate ſteps, to bring us to the knowledge of other quantities, which are real, intelligible, and required to be known. It is ſufficient that we arrive at them by a regular progreſs of diminution, and by a juſt and neceſſary way of reaſoning; and that they are afterwards duly eliminated, and leave us intelligible and indubitable Concluſions. For this will always be the conſequence, let the *media* of ratiocination be what they will, when we argue according to the ſtrict Rules of Art. And it is a very common thing in Geometry, to make impoſſible and abſurd Suppoſitions, which is the ſame thing as to introduce impoſſible quantities, and by their means to diſcover truth.

We have an inſtance ſimilar to this, in another ſpecies of Quantities, which, though as inconceiveable and as impoſſible as theſe can be, yet when they ariſe in Computations, they do not affect the Concluſion with their impoſſibility, except when they ought ſo to do; but when they are duly eliminated, by juſt Methods of Reduction, the Concluſion always remains ſound and good. Theſe Quantities are thoſe Quadratick Surds, which are diſtinguiſh'd by the name of impoſſible and imaginary Quantities; ſuch as $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-4}$, &c. For they import, that a quantity or number is to be found, which multiply'd by itſelf ſhall produce a negative quantity; which is manifeſtly impoſſible. And yet theſe quantities have all varieties of proportion to one another, as thoſe aforegoing are proportional to the poſſible and intelligible numbers 1; $\sqrt{2}$, $\sqrt{3}$, 2, &c. reſpectively; and when they ariſe in Computations, and are regularly eliminated and excluded, they always leave a juſt and good Concluſion.

Thus, for Example, if we had the Cubick Equation $x^3 - 12x^2 + 41x - 42 = 0$, from whence we were to extract the Root x; by proceeding according to Rule, we ſhould have this ſurd Expreſſion for the Root, $x = 4 + \sqrt[3]{3 + \sqrt{-\frac{100}{27}}} + \sqrt[3]{3 - \sqrt{-\frac{100}{27}}}$, in which the impoſſible quantity $\sqrt{-\frac{100}{27}}$ is involved; and yet this Expreſſion ought not to be rejected as abſurd and uſeleſs, becauſe, by a due Reduction, we may derive the true Roots of the Equation from it. For when the Cubick Root of the firſt *vinculum* is rightly extracted, it will be found to be the impoſſible Number $-1 + \sqrt{-\frac{4}{3}}$, as may appear by cubing; and when the Cubick Root of the ſecond *vinculum* is extracted, it will be found to be $-1 - \sqrt{-\frac{4}{3}}$. Then by collecting theſe Numbers, the impoſſible Number $\surd-\frac{4}{3}$ will be eliminated, and the Root of the Equation will be found $x=4-1-1=2$.

Or the Cubick Root of the firſt *vinculum* will alſo be $\frac{3}{2}+\surd-\frac{1}{12}$, as may likewiſe appear by Involution; and of the ſecond *vinculum* it will be $\frac{3}{2}-\surd-\frac{1}{12}$. So that another of the Roots of the given Equation will be $x=4+\frac{3}{2}+\frac{3}{2}=7$. Or the Cubick Root of the ſame firſt *vinculum* will be $-\frac{1}{2}-\surd-\frac{25}{12}$; and of the ſecond will be $-\frac{1}{2}+\surd-\frac{25}{12}$. So that the third Root of the given Equation will be $x=4-\frac{1}{2}-\frac{1}{2}=3$. And in like manner in all other Cubick Equations, when the ſurd *vincula* include an impoſſible quantity, by extracting the Cubick Roots, and then by collecting, the impoſſible parts will be excluded, and the three Roots of the Equation will be found, which will always be poſſible. But when the aforeſaid ſurd *vincula* do not include an impoſſible quantity, then by Extraction one poſſible Root only will be found, and an impoſſibility will affect the other two Roots, or will remain (as it ought) in the Concluſion. And a like judgment may be made of higher degrees of Equations.

So that theſe impoſſible quantities, in all theſe and many other inſtances that might be produced, are ſo far from infecting or deſtroying the truth of theſe Concluſions, that they are the neceſſary means and helps of diſcovering it. And why may we not conclude the ſame of that other ſpecies of impoſſible quantities, if they muſt needs be thought and call'd ſo? Surely it may be allow'd, that if theſe Moments and infinitely little Quantities are to be eſteem'd a kind of impoſſible Quantities, yet nevertheleſs they may be made uſeful, they may aſſiſt us, by a juſt way of Argumentation, in finding the Relations of Velocities, or Fluxions, or other poſſible Quantities required. And finally, being themſelves duly eliminated and excluded, they may leave us finite, poſſible, and intelligible Equations, or Relations of Quantities.

Therefore the admitting and retaining theſe Quantities, however impoſſible they may ſeem to be, the inveſtigating their Properties with our utmoſt induſtry, and applying thoſe Properties to uſe whenever occaſion offers, is only keeping within the Rules of Reaſon and Analogy; and is alſo following the Example of our ſagacious and illuſtrious Author, who of all others has the greateſt right to be our Precedent in theſe matters. 'Tis enlarging the number of general Principles and Methods, which will always greatly contribute to the Advancement of true Science. In ſhort, it will enable us to make a much greater progreſs and proficience, than we otherwiſe can do, in cultivating and improving what I have elſewhere call'd *The Philoſophy of Quantity.*

FINIS.

THE

CONTENTS of the following Comment.

I. *ANnotations on the Introduction; or the Reſolution of Equations by Infinite Series.* — pag. 143

Sect. I. *Of the nature and conſtruction of infinite or converging Series.* — p. 143

Sect. II. *The Reſolution of ſimple Equations, or of pure Powers, by infinite Series.* — p. 159

Sect. III. *The Reſolution of Numeral Affected Equations.* p. 186

Sect. IV. *The Reſolution of Specious Equations by infinite Series; and firſt for determining the forms of the Series, and their initial Approximations.* — p. 191

Sect. V. *The Reſolution of Affected ſpecious Equations proſecuted by various Methods of Analyſis.* — p. 209

Sect. VI. *Tranſition to the Method of Fluxions.* — p. 235

II. *Annotations on Prob.* 1. *or, the Relation of the flowing Quantities being given, to determine the Relation of their Fluxions.* — p. 241

Sect. I. *Concerning Fluxions of the firſt Order, and to find their Equations.* — p. 241

Sect. II. *Concerning Fluxions of ſuperior Orders, and the method of deriving their Equations.* — p. 257

Sect. III. *The Geometrical and Mechanical Elements of Fluxions.* — p. 266

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III. *Annotations on Prob.* 2. *or, the Relation of the Fluxions being given, to determine the Relation of the Fluents.* p.277

- Sect. I. *A particular Solution; with a preparation to the general Solution, by which it is diſtributed into three Caſes.* — — — p.277
- Sect. II. *Solution of the firſt Caſe of Equations.* — p.282
- Sect. III. *Solution of the ſecond Caſe of Equations.* — p.286
- Sect. IV. *Solution of the third Caſe of Equations, with ſome neceſſary Demonſtrations.* — — p.300
- Sect. V. *The Reſolution of Equations, whether Algebraical or Fluxional, by the aſſiſtance of ſuperior Orders of Fluxions.* — — — p.308
- Sect. VI. *An Analytical Appendix, explaining ſome Terms and Expreſſions in the foregoing Work.* — p.321
- Sect. VII. *The Concluſion; containing a ſhort recapitulation or review of the whole.* — — p.330

THE Reader is deſired to correct the following Errors, which I hope will be thought but few, and ſuch as in works of this kind are hardly to be avoided. 'Tis here neceſſary to take notice of even literal Miſtakes, which in ſubjects of this nature are often very material. That the Errors are ſo few, is owing to the kind aſſiſtance of a ſkilful Friend or two, who ſupply'd my frequent abſence from the Preſs; as alſo to the care of a diligent Printer.

ERRATA.

In the Preface, pag. xiii. *lin.* 3. *read* which is here ſubjoin'd. *Ibid.* l. 5. *for* matter *read* manner. Pag. xxiii. l. *ult.* *for* Preface, &c. *read* Concluſion of this Work. P. 7. l. 31. *for* + *read* =. P. 15. l. 9. *read* $y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4$, &c. P. 17. l. 17. *read* $-\frac{2}{9x}$. P. 32. l. 21. *read* $\frac{\dot y}{\dot x}$. P. 35. l. 3. *for* $10xy^4$ *read* $10x^4y$. P. 62. l. 4. *read* $\frac{\dot z + z^3}{\dot z}$. P. 63. l. 31. *for* y *read* $\dot y$. *Ibid.* l. *ult.* *for* $-\frac{yz}{v}$ *read* $-\frac{y\dot z}{v}$. P. 64. l. 9. *for* 2 *read* z. *Ibid.* l. 30. *read* $\dot t$. P. 82. l. 17. *read* $2\dot z z$. P. 87. l. 22. *read* $+2a^{\frac{3}{2}}x^{\frac{1}{2}}$. *Ibid.* l. 22, 24. *read* AFDB. P. 88. l. 27. *read* $\dot z$. P. 92. l. 5. *read* $+\frac{2a^2y^3}{5e^3}$. *Ibid.* l. 21. *for* z *read* $\dot z$. P. 104. l. 8. *read* $6ne^2$. P. 109. l. 33. *dele* as often. P. 110. l. 29. *read* $\sqrt{\frac{a^4}{a^2+z^2}} = \sqrt{AQ \times EC} = HA = AB = x$, and $\sqrt{a^2 - x^2} =$. P. 113. l. 17. *for* Parabola *read* Hyperbola. P. 119. l. 12. *read* $CE \times \dot{ID}$ = to the Fluxion of the Area ACEG, and $\dot{ID} \times IP$ = to. P. 131. l. 8. *read* $+\frac{ds}{2t}$. *Ibid.* l. 19. *read* $\frac{tt}{9ss}$. P. 135. l. 15. *read* Ab. P. 138. l. 9. *read* Abſciſs AB. P. 145. l. *penult.* *read* $7x^{-2}$. P. 149. l. 20. *read* which in. P. 157. l. 13. *read* ax. P. 168. l. 5. *read* $\frac{1}{2}ax$. P. 171. l. 17. *for* B^6 *read* β^6. P. 177. l. 15. *read* z^{m+2}. P. 189. l. 30. *for the laſt* $P-3$ *read* $P-4$. P. 204. l. 16. *read* to $2m$. P. 213. l. 7. *for* Species *read* Series. P. 229. l. 21. *for* x^{-3} *read* x^{-4}. *Ibid.* l. 24. *for* x^{-4} *read* x^{-3}. P. 234. l. 2. *for* yy *read* $\dot y$. P. 236. l. 26. *read* generating. P. 243. l. 29. *read* $-ax^2\dot y y^{-1}$. P. 284. l. *ult.* *read* $\frac{\dot y}{\dot x}$. P. 289. l. 17. *for* right *read* left. P. 295. l. 1, 2. *read* $-\frac{1}{6}x^4 + \frac{5}{12}ax^4$. P. 297. l. 19. *for* yx^{-1} *read* $\dot y x^{-1}$. P. 298. l. 14. *read* $-\dot y$. P. 304. l. 20, 21. *dele* $+bc$. P. 309. l. 18. *read* a^{m-3}. P. 317. l. *ult.* *read* $a^2\dot y^2$.

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